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Adaptive coupling of a stochastic continuum model with a deterministic one in the framework of the Arlequin method

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1 Presentation of authors

The work presented in this paper is part of Cédric Zaccardi's Ph.D., supervised by Hachmi Ben-Dhia (advisor), Régis Cottereau and Ludovic Chamoin (co-advisors) at *École Centrale Paris*. Cédric Zaccardi is a former student of the *École Normale Supérieure de Cachan* and holds an *aggrégation* in Mechanics and a M.Sc. in structural dynamics.

2 Context

For the last few decades, with the fast growth of computational performances, numerical simulations have provided an essential tool in engineering studies. Classical deterministic models are widely used and satisfactory for a large range of industrial applications. However, when one is interested in multiscale phenomena, local specific quantities, or local behaviors, these models are either too coarse or require too much information for the identification of their parameters. Stochastic methods have therefore been proposed.

Further, in many cases, local defects influence strongly the behavior of a structure in a localized region while the rest of the structure is only slightly modified. In these cases, it is neither reasonable nor tractable to model the structure entirely at a fine scale. Multiscale methods are thus appealing. Three major families can be distinguished in this domain: those based on homogenization techniques such as the FE^2, those based on the enrichment of the interested field (Partition of Unity Method, GFEM, X-FEM), and finally, those based on the superposition of fields or models (Variational Multiscale Method, Hierarchical Dirichlet Projection, S-Method...). In this last family, the Arlequin Method [1, 2] proposes a multi-model method with volume coupling. This method is applied in the case of a coupling between a deterministic continuum model and a stochastic one.

3 Coupling with the Arlequin Method

The Arlequin Method proposes a superposition model based on the following key points: (i) model superposition, (ii) volume coupling of the two models, (iii) distribution of the mechanical energy between the two models in the coupling zone.

By considering a complete probability space $(\Theta, \mathscr{T}, P)$ with Θ a set of outcomes, $\mathscr{T}$ a $\sigma-$algebra of events, and $P : \mathscr{T} \to [0,1]$ a probability measure, our Arlequin stochastic mechanical problem is written as follows:

Find (u_d, u_s, λ) in $V_d \times V_s \times V_c$ such as:

$$\left\{ \begin{array}{rcll} a_d(u_d, v) + C(\lambda, v) & = & \ell_d(v), & \forall v \in V_d \\ a_s(u_s, v) - C(\lambda, v) & = & \ell_s(v), & \forall w \in V_s \\ C(\mu, u_d - u_s) & = & 0, & \forall \mu \in V_c \end{array} \right. \tag{1}$$

where: $V_d = \{v \in \mathscr{H}^1(\Omega_d), v(x) = 0, \forall x \in \Gamma_u\}$, $V_s = \{v \in \mathscr{L}^2(\Theta, \mathscr{H}^1(\Omega_s))\}$. The coupling space V_c is built as a space composed of functions with a spatially varying mean and a perfectly spatially correlated randomness:

$$V_c = \{\psi(x) + \theta \mathbb{I}_c(x) | \psi \in \mathscr{H}^1(\Omega_c), \theta \in \mathscr{L}^2(\Theta, \mathbb{R}), \int_{\Omega_c} \psi(x)\, d\Omega = 0\} \tag{2}$$

where the indicator function $\mathbb{I}(x)$ is such that $\mathbb{I}_c(x \in \Omega_c) = 1$ and $\mathbb{I}_c(x \notin \Omega_c) = 0$. The coupling operator $C : V_c \times V_c \to \mathbb{R}$ is defined by:

$$C(u,v) = \mathrm{E}[\underline{C}(u,v)] = \mathrm{E}\left[\int_{\Omega_c} \kappa_0 uv + \kappa_1 \varepsilon(u) : \varepsilon(v)\, d\Omega\right]$$

where κ_0 and κ_1 are two constant coefficients used to ensure the consistency of dimensions.

The three equations defining (1) stand respectively for:

1. the equilibrium of the deterministic system defined in a subdomain Ω_d,
2. the equilibrium of the stochastic system defined in a subdomain Ω_s,
3. the coupling of the deterministic field u_d with the stochastic field u_s in a subdomain Ω_c of $S = \Omega_1 \cap \Omega_2 \neq 0$.

4 Goal-oriented error estimation

To control the quality of the approximate solution obtained with such an approach, a goal-oriented method [4] is introduced. To do so, the Arlequin problem (1) is written as follows:

Find (u_d, u_s, λ) in $V_d \times V_s \times V_c$ such as:

$$a((u_d, u_s, \lambda), (v, w, \mu)) = \ell(v, w) \qquad \forall (v, w, \mu) \in V_d \times V_s \times V_c \tag{3}$$

Following the idea developed in [5], an adaptive process is proposed to increase the accuracy of the model with respect to the computation of a specific quantity of interest (*e.g.* the mean of a component of the displacement field in a given time-space region). Denoting this quantity as $q(u_d, u_s)$ with (u_d, u_s) the Arlequin solution of (3), the adjoint problem is considered, it reads:

Find $(\tilde{p}_{u_d}, \tilde{p}_{u_s}, \tilde{p}_\lambda)$ in $\tilde{V}_d \times \tilde{V}_s \times \tilde{V}_c$ such as:

$$a'((\tilde{u}_d, \tilde{u}_s, \tilde{\lambda}); (v, w, \mu), (\tilde{p}_{u_d}, \tilde{p}_{u_s}, \tilde{p}_\lambda)) = q'((\tilde{u}_d, \tilde{u}_s); (v, w)) \qquad \forall (v, w, \mu) \in \tilde{V}_1 \times \tilde{V}_2 \times \tilde{V}_c \tag{4}$$

where " $'$ " denotes the Gâteaux derivatives and where the quantities denoted with a " $\sim$ " come from a coupling close to the previous one but with a larger coupling zone [5]. Finally, the assessment of this Arlequin formulation is performed using the residual of both problems (3-4).

References

[1] H. Ben Dhia, *Multiscale mechanical problems: the Arlequin method*, Comptes-Rendus de l'Académie des Sciences - séries IIB 326(12), 899-904, 1998.

[2] H. Ben Dhia, *Further insights by theorical investigations of the multiscale Arlequin method*, Int. J. Mult. Comp. Engrg. 6(3), 215-232, 2008.

[3] R. Cottereau, H. Ben Dhia, D. Clouteau, *Localized modeling of uncertainty in the Arlequin framework*, Proceedings of IUTAM Symposium on the Vibration Analysis of Structures with Uncertainties, R. Langley and A. Belyaev (eds), IUTAM Bookseries vol. 27, 477-488, 2010.

[4] J.T. Oden, S. Prudhomme, *Goal-oriented error estimation and adaptativity for the finite element method*, Comput. Math. Appl. 41, 735-756, 2001.

[5] S. Prudhomme, L. Chamoin, H. Ben Dhia, P. T. Bauman, *An adaptative strategy for the control of modeling error in two-dimensional atomic-to-continuum coupling simulations*, Comput. Methods Appl. Mech. Engrg. 197, 1887-1901, 2009.