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On Convergence Rates Equivalency and Sampling Strategies in Functional Deconvolution Models

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FORMULATION OF THE PROBLEM

PROBLEM: Estimate an unknown response function $f(\cdot)$ based on observations from the noisy convolutions

$$y(u,t) = \int_T f(t-x)g(u,x)dx + \frac{1}{\sqrt{n}} z(u,t), \quad t \in T = [0,1],\ u \in [a,b],$$

Here $y(u,t) = Y(u,t)/\sigma(u)$ and $g(u,x) = G(u,x)/\sigma(u)$.
Consequently, without loss of generality, we consider only the case when

$$\sigma(u) \equiv 1$$

DISCRETIZATION

Consider a discretization of the functional deconvolution model when $y(u,t)$ is observed at $n = NM$ points

$$(u_l, t_i), l = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, N.$$

Equation takes the form

$$y(u_l, t_i) = \int_T f(t_i - x) g(u_l, x) dx + \varepsilon_{il}, \quad t_i \in T = [0, 1], \ u_l \in [a, b],$$

ε_{il} are standard Gaussian random variables,

ε_{il} are independent for different i and l.

MOTIVATION

Continuous model can be viewed as a generalization of a multitude of **inverse problems in mathematical physics** where one needs to **recover initial or boundary conditions** on the basis of observations of a noisy solution of a partial differential equation (Lattes & Lions (1967), Golubev & Khasminskii (1999), Hesse (2007)).
However, in those problems, **in real life, observations can be made only at some particular points** $u_l, \ l = 1, 2, \ldots, M$.

Discrete model can be viewed as a a generalization of **multichannel deconvolution problem** (Casey & Walnut (1994), Pensky & Zayed (2002), De Canditiis & Pensky (2004, 2006))
Difference: the number of channels M can turn to infinity

CONSTRUCTION OF THE ESTIMATORS

Let $\varphi_{jk}(x)$ and $\psi_{jk}(x)$ be the periodized version of Meyer scaling and mother wavelet functions obtained as in Johnstone *et al.* (2004). Denote **the inner product in the Hilbert space $L^2(T)$ by** $\langle \cdot, \cdot \rangle$ and $e_m(t) = e^{i2\pi mt}$.

Denote the Fourier coefficients of $\varphi_{jk}(\cdot)$, $\psi_{jk}(\cdot)$ and $f(\cdot)$ by

$$\varphi_{mj_0k} = \langle e_m, \varphi_{j_0k} \rangle, \quad \psi_{mjk} = \langle e_m, \psi_{jk} \rangle, \quad f_m = \langle e_m, f \rangle.$$

For each $u \in [a, b]$, denote the functional Fourier coefficients by

$$y_m(u) = \langle e_m, y(u, \cdot) \rangle, \quad g_m(u) = \langle e_m, g(u, \cdot) \rangle, \quad z_m(u) = \langle e_m, z(u, \cdot) \rangle.$$

ESTIMATING FOURIER COEFFICIENTS: CONTINUOUS CASE

By properties of the Fourier transform, we have

$$y_m(u) = g_m(u) f_m + n^{-1/2} z_m(u), \quad u \in [a, b],$$

where $z_m(u)$ are Gaussian processes with zero mean and covariance function

$$\mathbb{E}[z_{m_1}(u_1) z_{m_2}(u_2)] = \delta(m_1 - m_2)\delta(u_1 - u_2).$$

Multiply both sides of the equation by $\overline{g_m(u)}$ and integrate over $u \in [a, b]$. Obtain

$$\widehat{f}_m = \left(\int_a^b \overline{g_m(u)} y_m(u) du \right) \Big/ \left(\int_a^b |g_m(u)|^2 du \right).$$

When $a = b$ the estimator $\widehat{f}_m$ takes the form

$$\widehat{f}_m = \overline{g_m(a)} y_m(a) \big/ |g_m(a)|^2.$$

ESTIMATING FOURIER COEFFICIENTS DISCRETE CASE

Using properties of the discrete Fourier transform, derive

$$y_m(u_l) = g_m(u_l) f_m + N^{-1/2} z_{ml}, \quad l = 1, 2, \ldots, M,$$

where z_{ml} are Gaussian random variables with zero mean and covariance function

$$\mathbb{E}(z_{m_1 l_1} z_{m_2 l_2}) = \delta(m_1 - m_2)\,\delta(l_1 - l_2).$$

Similarly to the continuous case, multiply both sides of the equation by $\overline{g_m(u_l)}$ and add them together to obtain

$$\widehat{f}_m = \left(\sum_{l=1}^{M} \overline{g_m(u_l)} y_m(u_l) \right) \Big/ \left(\sum_{l=1}^{M} |g_m(u_l)|^2 \right).$$

Here, and in what follows, we abuse notation and f_m refers to both **functional Fourier coefficients and their discrete counterparts**.

ESTIMATING WAVELET COEFFICIENTS

By Plancherel's formula, the scaling and wavelet coefficients a_{j_0k} and b_{jk}, respectively, of f can be represented as

$$a_{j_0k} = \sum_{m\in C_{j_0}} f_m \varphi_{mj_0k}, \quad b_{jk} = \sum_{m\in C_j} f_m \psi_{mjk},$$

where, due to the fact that Meyer wavelets are band limited,

$$C_j = \left\{m : \psi_{mjk} \neq 0\right\} \subset 2\pi/3[-2^{j+2}, -2^j] \cup [2^j, 2^{j+2}].$$

Substitute f_m by $\widehat{f}_m$ and obtain

$$\widehat{a}_{j_0k} = \sum_{m\in C_{j_0}} \widehat{f}_m \varphi_{mj_0k}, \quad \widehat{b}_{jk} = \sum_{m\in C_j} \widehat{f}_m \psi_{mjk}.$$

BLOCK THRESHOLDING WAVELET ESTIMATORS

Divide the wavelet coefficients at each resolution level into blocks of length $\ln n$. Denote

$$\begin{aligned} A_j &= \left\{ r \mid r = 1, 2, \ldots, 2^j / \ln n \right\}, \\ U_{jr} &= \left\{ k \mid k = 0, 1, \ldots, 2^j - 1;\ (r-1)\ln n \le k \le r \ln n - 1 \right\}, \\ \widehat{B}_{jr} &= \sum_{k \in U_{jr}} \widehat{b}_{jk}^2. \end{aligned}$$

Finally, reconstruct $f(t)$ as

$$\widehat{f}_n(t) = \sum_{k=0}^{2^{j_0}-1} \widehat{a}_{j_0 k} \varphi_{j_0 k}(t) + \sum_{j=j_0}^{J-1} \sum_{r \in A_j} \sum_{k \in U_{jr}} \widehat{b}_{jk} \mathbb{I}(|\widehat{B}_{jr}| > \lambda_j)\, \psi_{jk}(t),$$

where resolution levels j_0 and J and the thresholds λ_j will be defined later.

THE RISK: NOTATIONS

For an r-regular multiresolution analysis with $0 < s < r$, denote by $B^s_{p,q}(A)$ a Besov ball $B^s_{p,q}(A)$ of radius $A > 0$ with $1 \le p, q \le \infty$. Denote by $\|g\|$ the L^2-norm of a function $g(\cdot)$ and

$$s^* = s + 1/2 - 1/p', \quad p' = \min(p, 2),$$

Denote $\underline{u} = (u_1, u_2, \ldots, u_M)$ and

$$\tau_1^c(m) = \int_a^b |g_m(u)|^2\, du \quad \text{and} \quad \tau_1^d(m, \underline{u}, M) = \frac{1}{M} \sum_{l=1}^{M} |g_m(u_l)|^2.$$

The minimax risks of the estimators in the continuous and discrete cases are determined by $\tau_1^c(m)$ and $\tau_1^d(m, \underline{u}, M)$, respectively.

MINIMAX LOWER BOUNDS FOR THE RISK

Define **the minimax L^2-risks** over the set Ω as

$$R_n^c(\Omega) = \inf_{\tilde{f}_n^c} \sup_{f\in\Omega} \mathbb{E}\|\tilde{f}_n^c - f\|^2, \quad \text{continuous}$$

$$R_n^d(\Omega, \underline{u}, M) = \inf_{\tilde{f}_n^d} \sup_{f\in\Omega} \mathbb{E}\|\tilde{f}_n^d - f\|^2, \quad \text{discrete, fixed points}$$

$$R_n^d(\Omega) = \inf_{\underline{u},M} R_n^d(\Omega, \underline{u}, M). \quad \text{discrete, total minimum}$$

Let,

in the continuous case

$\tau_1(m) = \tau_1^c(m)$, $R_n^*(B_{p,q}^s(A)) = R_n^c(B_{p,q}^s(A))$

in the discrete case

$\tau_1(m) = \tau_1^d(m, \underline{u}, M)$, $R_n^*(B_{p,q}^s(A)) = R_n^d(B_{p,q}^s(A), \underline{u}, M)$.

MINIMAX LOWER BOUNDS FOR THE RISK

Assumption: for some constants $\nu, \lambda \in \mathbb{R}$, $\alpha \geq 0$, $\beta > 0$ and $K_1 > 0$, **independent** of m and n, but **possibly dependent** on M and $\underline{u}$,

$$\tau_1(m) \leq K\varepsilon_n \, |m|^{-2\nu} (\ln |m|)^{-\lambda} \exp\Big(-\alpha |m|^{\beta}\Big), \quad \nu > 0 \quad \text{if} \quad \alpha = 0.$$

Denote $n^* = n\varepsilon_n$ and assume that ε_n are such that

$$n^* = n\varepsilon_n \to \infty \quad \text{as} \quad n \to \infty.$$

Then, as $n \to \infty$,

$$R_n^*(B_{p,q}^s(A)) \geq \begin{cases} C n^{-\frac{2s}{2s+2\nu+1}} (\ln n)^{\frac{2s\lambda}{2s+2\nu+1}}, & \text{if} \quad \alpha = 0, \ \nu(2-p) < ps^*, \\ C\left(\frac{\ln n}{n}\right)^{\frac{2s^*}{2s^*+2\nu}} (\ln n)^{\frac{2s^*\lambda}{2s^*+2\nu}}, & \text{if} \quad \alpha = 0, \ \nu(2-p) \geq ps^*, \\ C(\ln n)^{-\frac{2s^*}{\beta}}, & \text{if} \quad \alpha > 0. \end{cases}$$

MINIMAX UPPER BOUNDS FOR THE RISK

Assumption: for some constants $\nu, \lambda \in \mathbb{R}$, $\alpha \geq 0$, $\beta > 0$ and $K_1 > 0$, **independent** of m and n, but **possibly dependent** on M and $\underline{u}$, Let, as before, $n^* = n\varepsilon_n \to \infty$ and, in the case of $\alpha = 0$, the sequence ε_n be such that

$$-h_1 \ln n \leq \ln(1/\varepsilon_n) \leq (1 - h_2) \ln n$$

for some constants $h_1 > 0$ and $h_2 \in (0, 1)$. Then for $R_n = \sup_{f \in B^s_{p,q}(A)} \mathbb{E}\|\widehat{f}_n - f\|^2$ one has

$$R_n \leq \begin{cases} C(n^*)^{-\frac{2s}{2s+2\nu+1}} (\ln n)^{\varrho + \frac{2s\lambda}{2s+2\nu+1}}, & \text{if} \quad \alpha = 0,\ \nu(2-p) < ps^*, \\ C\left(\frac{\ln n}{n^*}\right)^{\frac{2s^*}{2s^*+2\nu}} (\ln n)^{\varrho + \frac{2s^*\lambda}{2s^*+2\nu}}, & \text{if} \quad \alpha = 0,\ \nu(2-p) \geq ps^*, \\ C(\ln(n^*))^{-\frac{2s^*}{\beta}}, & \text{if} \quad \alpha > 0. \end{cases}$$

THE INTERPLAY BETWEEN CONTINUOUS AND DISCRETE MODELS

The convergence rates depend on two aspects:

a) the total number of observations $n = NM$;

b) the behavior of $\tau_1(m)$.

In the continuous model, the values of $\tau_1^c(m)$ depend on m only.
In the discrete model, the values of $\tau_1^d(m, \underline{u}, M)$ depend on m and may depend on M and observation points $\underline{u}$.

If the values of $\tau_1^d(m, \underline{u}, M)$ are independent of the choice of M and $\underline{u}$, then **the convergence rates in the discrete and the continuous models coincide**. Moreover, in this case, **the wavelet estimator is asymptotically optimal** (in the minimax sense), no matter what the choice of M is.

CONVERGENCE RATES EQUIVALENCY BETWEEN DISCRETE AND CONTINUOS MODELS

Assume that there exist points $u_*, u^* \in [a,b]$, independent of m, such that, for any $u \in [a,b]$,

$$|g_m(u)| \leq K|g_m(u^*)| \text{ and } |g_m(u)| \geq K|g_m(u_*)|.$$

Necessary and sufficient conditions for convergence rates equivalency. Let there exist constants $\nu_1 \in \mathbb{R}$, $\nu_2 \in \mathbb{R}$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_1 > 0$ and $\beta_2 > 0$, independent of m and n, such that

$$|g_m(u^*)|^2 \asymp |m|^{-2\nu_1} \exp(-\alpha_1 |m|^{\beta_1}), \quad \nu_1 > 0 \quad \text{if} \quad \alpha_1 = 0,$$
$$|g_m(u_*)|^2 \asymp |m|^{-2\nu_2} \exp(-\alpha_2 |m|^{\beta_2}), \quad \nu_2 > 0 \quad \text{if} \quad \alpha_2 = 0.$$

Then, the convergence rates in the discrete model are independent of the choice of M and the selection of points $\underline{u}$, and, hence, coincide with the convergence rates in the continuous model, if and only if

$$\alpha_1\alpha_2 > 0 \text{ and } \beta_1 = \beta_2 \quad \textbf{or} \quad \alpha_1 = \alpha_2 = 0 \text{ and } \nu_1 = \nu_2.$$

CONDITIONS FOR EQUI-RATES

Conditions above can be divided into the following two groups:

Condition I. There exist constants $\nu_1 \in \mathbb{R}$, $\alpha_1 \geq 0$ and $\beta_1 > 0$ and a point $u^* \in [a, b]$, independent of m and n, such that

$$|g_m(u)|^2 \leq K|g_m(u^*)|^2 \asymp |m|^{-2\nu_1} \exp(-\alpha_1|m|^{\beta_1}), \quad \nu_1 > 0 \text{ if } \alpha_1 = 0.$$

Condition I*. There exist constants $\nu_2 \in \mathbb{R}$, $\alpha_2 \geq 0$ and $\beta_2 > 0$, and a point $u_* \in [a, b]$, independent of m and n, such that

$$|g_m(u)|^2 \geq K|g_m(u_*)|^2 \asymp |m|^{-2\nu_2} \exp(-\alpha_2|m|^{\beta_2}), \quad \nu_2 > 0 \text{ if } \alpha_2 = 0.$$

Condition II. Either $\alpha_1\alpha_2 > 0$ and $\beta_1 = \beta_2$ or $\alpha_1 = \alpha_2 = 0$ and $\nu_1 = \nu_2$.

SOME EXAMPLES

The following inverse Mathematical Physics problems can be reformulated as a functional deconvolution problem.

Example 1: Estimation of the initial condition in the heat conductivity equation.

Let $h(t,x)$ be a solution of the heat conductivity equation

$$\frac{\partial h(t,x)}{\partial t} = \frac{\partial^2 h(t,x)}{\partial x^2}, \quad x \in [0,1], \ t \in [a,b], \ a > 0, \, b < \infty,$$

with initial condition $h(0,x) = f(x)$ and periodic boundary conditions $h(t,0) = h(t,1)$ and $\partial h(t,x)/\partial x|_{x=0} = \partial h(t,x)/\partial x|_{x=1}$.

A noisy solution $y(t,x) = h(t,x) + n^{-1/2} z(t,x)$ is observed This problem was considered by Lattes & Lions (1967) and further studied by Golubev & Khasminskii (1999). One has

$$g_m(u) = \exp(-4\pi^2 m^2 u).$$

Then, $u_* = b$, $u^* = a$, $|g_m(u_*)| = \exp(-4\pi^2 b m^2)$ and $|g_m(u^*)| = \exp(-4\pi^2 a m^2)$.

Hence, **Conditions I, I* and II hold** with $\nu_1 = \nu_2 = 0$, $\alpha_1 = 4\pi^2 b$, $\alpha_2 = 4\pi^2 a$ and $\beta_1 = \beta_2 = 2$.

Example 2: Estimation of the boundary condition for the Dirichlet problem of the Laplacian on the unit circle

Let $h(x,w)$ be a solution of the Dirichlet problem of the Laplacian on a region D on the plane

$$\frac{\partial^2 h(x,w)}{\partial x^2} + \frac{\partial^2 h(x,w)}{\partial w^2} = 0, \quad (x,w) \in D \subseteq \mathbb{R}^2,$$

with a boundary ∂D and boundary condition $h(x,w)\Big|_{\partial D} = F(x,w)$ where D is the unit circle.

Rewrite the function $h(\cdot,\cdot)$ in polar coordinates: $h(x,w) = h(u,t)$, where $u \in [0,1]$ is the polar radius and $t \in [0,2\pi]$ is the polar angle.

Suppose that only a noisy version $y(u,t) = h(u,t) + n^{-1/2} z(u,t)$ is observed. This problem was investigated in Golubev & Khasminskii (1999) and Golubev (2004).

Example 2: Estimation of the boundary condition for the Dirichlet problem of the Laplacian on the unit circle: continuation

One has

$$|g_m(u)| = K u^{|m|} = K \exp(-|m| \ln(1/u)) \quad u \in [0, r_0],$$

so that $u_* = 0$, $u^* = r_0$, $|g_m(u^*)| = K \exp(-|m| \ln(1/r_0))$ and $|g_m(u_*)| = 0$.

Then, **Condition I holds but Conditions I* and II do not hold** since $|g_m(u_*)| = 0$.

Hence, one cannot be certain that the convergence rates in the continuous and the discrete models coincide for any sampling scheme. Actually, **if sampling is carried out entirely at the single point** $u_* = 0$, then $\tau_1^d(m, u_*, 1) = 0$ and **we cannot recover the boundary condition** $f(\cdot)$.

Example 3: Estimation of the speed of a wave on a finite interval.

Let $h(t,x)$ be a solution of the wave equation

$$\frac{\partial^2 h(t,x)}{\partial t^2} = \frac{\partial^2 h(t,x)}{\partial x^2}$$

with initial–boundary conditions $h(0,x) = 0$, $h(t,0) = h(t,1) = 0$ and $\partial h(t,x)/\partial t\Big|_{t=0} = f(x)$, $x \in [0,1]$. The goal is to recover the speed of a wave $f(\cdot)$ on the basis of observing a noisy solution $y(t,x) = h(t,x) + n^{-1/2} z(t,x)$, where $t \in [a,b]$, $a > 0$, $b < 1$. One has

$$g_m(u) = (2\pi m)^{-1} \, \sin(2\pi m u), \; m \in \mathbb{Z} \setminus \{0\}, \; u \in [a,b].$$

None of Conditions I, I* and II hold.
The points u_* and u^* depend on m, hence, the convergence rates depend on the selection of M and $\underline{u}$. Actually, **if** $M = 1$ **and** u **is an integer**, then $\tau_1^d(m,u,1) = 0$ and **we cannot recover the speed of a wave** $f(\cdot)$.

POSSIBLE CASES

Consider now the following three cases.

1. **The uniform case**: **Conditions I, I* and II hold.**
Example: Estimation of the initial condition in the heat conductivity equation (Example 1)

2. **The regular case**: **Condition I holds but Condition II does not hold. Condition I* holds or, possibly,** $|g_m(u_*)| = 0$.
Example: Estimation of the boundary condition for the Dirichlet problem of the Laplacian on the unit circle (Example 2)

3. **The irregular case**: **Condition I does not hold.**
Example: Estimation of the speed of a wave on a finite interval (Example 3)

THE REGULAR CASE

Let Condition I hold:

$|g_m(u)|^2 \leq K|g_m(u^*)|^2 \asymp |m|^{-2\nu_1} \exp(-\alpha_1|m|^{\beta_1}), \;\; \nu_1 > 0$ if $\alpha_1 =$

Then

$$R_n^c(B_{p,q}^s(A)) \geq C R_n^d(B_{p,q}^s(A), u^*, 1) \asymp R_n^d(B_{p,q}^s(A)).$$

Also, for any choice of M and $\underline{u}$, we have

$$\sup_{f \in B_{p,q}^s(A)} \mathbb{E}\|\hat{f}_n^{d*} - f\|^2 \;\leq\; C \sup_{f \in B_{p,q}^s(A)} \mathbb{E}\|\hat{f}_n^c - f\|^2,$$
$$\sup_{f \in B_{p,q}^s(A)} \mathbb{E}\|\hat{f}_n^{d*} - f\|^2 \;\leq\; C \sup_{f \in B_{p,q}^s(A)} \mathbb{E}\|\hat{f}_n^d - f\|^2.$$

Conclusion: sampling **entirely at the single point** u^* leads to **the highest possible convergence rates in the discrete model** which are, possibly, higher than in continuous model.

PSEUDO-UNIFORM SAMPLING STRATEGIES

When the discrete model is replaced by the continuous model, **the underlying implicit assumption is that sampling is carried out at $M = M_n$ equidistant points with** $M_n \to \infty$.

Study an extension of this sampling scheme. In order to accommodate various sampling strategies, we consider a continuously differentiable function $S(x)$, $x \in [0,1]$, such that

$$0 \le s_1 \le S'(x) \le s_2 < \infty, \quad S(0) = a, \; S(1) = b.$$

Example: $S(x) = a + (b-a)x^h$, where $0 < h < \infty$ (the case $h = 1$ corresponds to the uniform sampling).

Let $d \in [0,1]$ and let consider the sample

$$u_l = S\left(\frac{l-1+d}{M}\right), \quad l = 1, 2, \ldots, M.$$

RATE CONVERGENCE EQUIVALENCY: ASSUMPTIONS FOR QUASI-UNIFORM SAMPLING

Let $g_m(u)$ satisfy the assumption

$$|g_m(u)|^2 \asymp |m|^{-2\nu(u)} \exp\Big(-\alpha(u)|m|^{\beta(u)}\Big), \quad u \in U,$$

for some continuous functions $\nu(u)$, $\alpha(u)$ and $\beta(u)$, $u \in U$, such that either $\alpha(u) = 0$ and $\nu(u) > 0$ or $\alpha(u) > 0$ and $\beta(u) > 0$, for all $u \in U$. Denote

$$u^* = \begin{cases} \arg\min_{u\in U} \nu(u), & \text{if} \quad \alpha(u) \equiv 0, \\ \arg\min_{u\in U} \beta(u), & \text{if} \quad \alpha(u) > 0, \ \beta(u) \neq \text{const}. \end{cases}$$

Assume that, in the neighborhood of point $u = u^*$, the function $\beta(\cdot)$ is continuously differentiable (if $\alpha(u) > 0$, $u \in U$) or the function $\nu(\cdot)$ is k-times continuously differentiable (if $\alpha(u) = 0$, $u \in U$), where $k \geq 1$ is such that

$$\nu^{(s)}(u^*) = 0, \ \ s = 1, \ldots, k-1, \ \ \nu^{(k)}(u^*) \neq 0,$$

with $\nu^{(s)}(\cdot)$ denoting the s-th derivative of the function $\nu(\cdot)$.

RATE CONVERGENCE EQUIVALENCY: QUASI-UNIFORM SAMPLING

The convergence rates in the discrete and the continuous models coincide up to, at most, a logarithmic factor if

$$\alpha(u) \equiv 0 \quad \text{and} \quad \lim_{n\to\infty} M_n^{-1} \ln n = \tau_1 < \infty,$$

or

$$\alpha(u) > 0 \quad \text{and} \quad \lim_{n\to\infty} M_n^{-1} \ln \ln n = \tau_2 < \infty.$$

If, moreover, $|g_m(u)|^2 = K|m|^{-2\nu(u)}$ for some continuously differentiable function $\nu(u)$, $u \in U$, and also

$$\lim_{n\to\infty} M_n^{-1} \, (\ln n)^{1+1/k} = 0,$$

where k is such that $\nu^{(k)}(u^*) \neq 0$, then **the convergence rates in the discrete and the continuous models coincide up to a constant.**

EXAMPLE 2: CONTINUATION

Recall that $|g_m(u)|^2 \asymp \exp(-2\ln(1/u)|m|)$, $u \in [0, r_0]$, so that $\beta = 1$, $\alpha(u) = 2\ln(1/u)$ and $u^* = r_0$. If the discrete model **is sampled entirely at the single point** u^*, then **the convergence rates in the continuous and the discrete models coincide.**

In the case of **the pseudo-uniform sampling**, the convergence rates in the discrete and the continuous models **coincide for any value of** M. This follows from the fact that, for any M, one has

$$
\begin{aligned}
\tau_1^d(m, \underline{u}, M) &\leq r_0^{2|m|}. \\
\tau_1^d(m, \underline{u}, M) &\geq 0.5 \exp\left(-2|m| \log(2/s_1)\right).
\end{aligned}
$$

EXAMPLE 4: SAMPLING AT THE BEST POINT

Let $g_m(u)$ satisfy

$$|g_m(u)|^2 \asymp \exp\left(-\alpha|m|^u\right), \quad 0 < a \le u \le b < \infty,$$

for some constant $\alpha > 0$, independent of m. Then $u^* = a$,

$$\begin{aligned} \tau_1^d(m, u^*, 1) &= |g_m(u^*)|^2 \asymp \exp\left(-\alpha|m|^a\right) \\ \tau_1^c(m) &\ge C|m|^{-b}(\ln|m|)^{-1} \exp(-\alpha|m|^a). \end{aligned}$$

Hence, **the convergence rates in the continuous and the discrete models coincide if sampling is carried out entirely at the single point** u^*.

EXAMPLE 4: PSEUDO-UNIFORM SAMPLING

Let $d \in [0,1]$ and $u_l = S\left(\frac{l-1+d}{M}\right), \quad l = 1, 2, \ldots, M.$

If $d > 0$, then **condition $\lim_{n\to\infty} M_n^{-1} \ln\ln n = \tau_2 < \infty$ is necessary in order the convergence rates in the discrete and the continuous models to coincide up to at most a constant.** Then, as $n \to \infty$,

$$R_n^c(B_{p,q}^s(A)) \asymp (\ln n)^{-\frac{2s^*}{a}},$$

$$R_n^d(B_{p,q}^s(A), \underline{u}, M_n) \asymp (\ln n)^{-\frac{2s^*}{a+d/M_n}},$$

so that

$$\lim_{n\to\infty} R_n^d(B_{p,q}^s(A), \underline{u}, M_n) \Big/ R_n^c(B_{p,q}^s(A)) = \infty$$

if $M = M_n$ is such that $\lim_{n\to\infty} M_n^{-1} \ln\ln n = \infty$.

EXAMPLE 5: SAMPLING AT THE BEST POINT

Let the functional Fourier coefficients $g_m(u)$ satisfy

$$|g_m(u)|^2 \asymp |m|^{-2\nu} \exp(-u|m|^\beta), \quad 0 \le u \le b < \infty,$$

for some constants $\nu > 0$ and $\beta > 0$, independent of m. Then, $u^* = 0$ and

$$\tau_1^d(m, u^*, 1) \asymp |g_m(u^*)|^2 \asymp |m|^{-2\nu}.$$

On the other hand, it is easy to check that

$$\tau_1^c(m) \asymp |m|^{-2\nu} \int_0^b \exp(-u|m|^\beta) du \asymp |m|^{-(2\nu+\beta)}.$$

Hence, **when sampling is carried out entirely at the single point $u^* = 0$, the convergence rates in the continuous model are worse than in the discrete model.**

EXAMPLE 5: SAMPLING AT THE BEST POINT, CONTINUATION

In particular,

$$R_n^c(B_{p,q}^s(A)) \geq \begin{cases} Cn^{-\frac{2s}{2s+2\nu+\beta+1}}, & \text{if} \quad \nu(2-p) < ps^*, \\ C\left(\frac{\ln n}{n}\right)^{\frac{2s^*}{2s^*+2\nu+\beta}}, & \text{if} \quad \nu(2-p) \geq ps^*, \end{cases}$$

and

$$R_n^d(B_{p,q}^s(A)) \asymp R_n^d(B_{p,q}^s(A), u^*, 1) \geq \begin{cases} Cn^{-\frac{2s}{2s+2\nu+1}}, & \text{if} \quad \nu(2-p) < ps^*, \\ C\left(\frac{\ln n}{n}\right)^{\frac{2s^*}{2s^*+2\nu}}, & \text{if} \quad \nu(2-p) \geq ps^*, \end{cases}$$

Hence, the convergence rates, in both discrete and continuous models, are polynomial, and the convergence rates are inferior in the continuous model.

EXAMPLE 5: PSEUDO-UNIFORM SAMPLING

For the pseudo-uniform sampling, one obtains, by direct calculations, that

$$\frac{K\,|m|^{-2\nu}e^{-s_2 d|m|^\beta/M}}{M\left(1-e^{-s_2|m|^\beta/M}\right)} \le \tau_1^d(m,\underline{u},M) \le \frac{K\,|m|^{-2\nu}e^{-s_1 d|m|^\beta/M}}{M\left(1-e^{-s_1|m|^\beta/M}\right)}.$$

Therefore, **the convergence rates in the discrete model depend on the value of d and the asymptotic behavior of $|m|^\beta/M_n$.**

1. **If M_n is large**, so that $|m|^\beta/M_n \to 0$ as $n \to \infty$, then

$$\tau_1^d(m,\underline{u},M_n) \asymp |m|^{-(2\nu+\beta)}$$

and the convergence rates in the discrete and the continuous models **coincide.**

2. **If** $M = M_n$ **is finite and** $d > 0$, then the convergence rates in the discrete model are **logarithmic**

$$R_n^d(B_{p,q}^s(A), \underline{u}, M) \geq C(\ln n)^{-\frac{2s^*}{\beta}}.$$

and **are inferior** to the **polynomial** convergence rates in the continuous model.

3. **If** $M = M_n$ **is finite and** $d = 0$, then the convergence rates in the discrete model coincide with $R_n^d(B_{p,q}^s(A))$ and are **superior** to those in the continuous model.

4. **If** M_n **is small**, so that $|m|^\beta/M_n \to \infty$ as $n \to \infty$, then

$$K\, M_n^{-1}\, |m|^{-2\nu} e^{-s_2 d|m|^\beta/M_n} \leq \tau_1^d(m, \underline{u}, M_n) \leq K\, M_n^{-1}\, |m|^{-2\nu} e^{-s_1 d|m|^\beta/M_n}$$

The convergence rates in the discrete model are **inferior** to the convergence rates in the continuous model (the difference depends on the rate of $|m_n|^\beta/M_n$ as $n \to \infty$). One can obtain **convergence rates in between** $R_n^d(B_{p,q}^s(A))$ and $C(\ln n)^{-\frac{2s^*}{\beta}}$.

IRREGULAR CASE: BOX-CAR BLURRING FUNCTION

For any $\underline{u}$

$$\tau_1^d(m, \underline{u}, M) \leq K m^{-2}.$$

Hence, **the rates of convergence in discrete model cannot be higher than in continuous model**.

Note, $\tau_1^d(m, \underline{u}, M)$ **is not bounded from below by a quantity independent** from M and points u_l, $l = 1, 2, \ldots, M$.

For example, if $M = 1$ and u_1 is an integer, $\tau_1^d(m, u_1, 1) = 0$.

Hence, **the choice** of M and **the selection of points** u_l, $l = 1, 2, \ldots, M$, **really matter**.

SOME BACKGROUND IN NUMBER THEORY

An irrational number a is called **Badly Approximable (BA)** if $\sup_n a_n < \infty$. If a is A **BA**, then for some constant $C(a)$

$$\inf \{\|ka\| : \ 1 \leq k \leq q\} \geq B(a)/q.$$

The notion of a BA number can be extended to an **M-tuple**. We say that **M-tuple** of irrational numbers $a_1, a_2, \cdots, a_M$ is **Badly Approximable (BA)** if

$$\max_i \inf_{1 \leq k \leq q} (\|ka_i\|) \geq Bq^{-1/M}.$$

BOX-CAR DISCRETE MODEL: FIXED M

Recall : $\nu = 1$ **in the continuous model**.

$M = 1$: If u_1 is a **BA irrational number**, then $\nu = 3/2$ (Johnstone, Kerkyacharian, Picard & Raimondo (2004)).

$M \geq 2$, **fixed**: If one of the u_l's is a **BA irrational number**, and $u_1, u_2, \ldots, u_M$ is a **BA M -tuple**, then $\nu = 1 + 1/(2M)$ in the upper bound of the risk (De Canditiis & Pensky (2006)).

$M = M_n \to \infty$ **as** $n \to \infty$: require **non-trivial results in number theory**.

BOX-CAR: EQUI-RATES

Recall that $\tau_1^d(m, \underline{u}, M) \le Km^{-2}$ for any $\underline{u}$.

If $M = M_n \to \infty$ **fast enough**, then an appropriate selection of points u_l, $l = 1, 2, \ldots, M$, can secure an opposite inequality, so that **the rates of convergence in the discrete and the continuous models are equal**.

If $M \ge M_{0n} = (32\pi/3)(b-a)n^{1/3}$ and $u_l = a + (b-a)l/M$, $l = 1, 2, \ldots, M$. Then for n and $|m|$ large enough,

$$\tau_1^d(m, \underline{u}, M) \ge Km^{-2},$$

and **the asymptotical minimax rates of convergence for the risk coincide in the discrete and the continuous models**.

BOX-CAR: SMALLER VALUES OF M

If $M_n \to \infty$ but **at a slow rate**, one has to **employ a BA M-tuple**.

Since in $\max_i \inf_{1\le k\le q}(\|ka_i\|) \ge Bq^{-1/M}$ the constant B depends on M, i.e.$B = B(M)$, results for finite M **cannot be generalized automatically**.

In addition, one **needs a procedure for construction a BA M-tuple** in a specified interval $[a, b]$.

BA M-TUPLE

We **suggested the algorithm for construction of a BM-tuple** $\beta_1, \beta_2, \ldots, \beta_M$ of **increasing length** M on an arbitrary interval (a, b), of a **non-asymptotic length.**

We showed that, $M \to \infty$, one has

$$\max_{i=1,2,\ldots,M} |\beta_i q - p_i| \geq B_0 \exp(-6M \ln M)\, q^{-1/M},$$

for any integer numbers $q > 0$ and $p_1, p_2, \ldots, p_M$, and for some constant $B_0 > 0$, independent of M, q and $p_1, p_2, \ldots, p_M$, so that

$$B(M) = B_0 \exp(-6M \ln M).$$

BOX-CAR: CONVERGENCE RATES

Let $(\beta_1, \beta_2, \ldots, \beta_M)$ be a BA M-tuple constructed on the interval $(2a, 2b)$ and let one of $\beta_1, \beta_2, \ldots, \beta_M$ be a BA number. Choose $u_l = \beta_l/2$, $l = 1, 2, \ldots, M$, and

$$M = M_n = \nu \sqrt{\ln n/(\ln \ln n)}$$

for some $\nu \le 1/\sqrt{6}$, independent of n. Then, as $n \to \infty$,

$$R_n^d(B_{p,q}^s(A), \underline{u}, M_n) \le \begin{cases} C\, n^{-\frac{2s}{2s+3}}\, \Delta_n, & \text{if } \; s > 3(1/p - 1/2), \\ C \left(\frac{\ln n}{n}\right)^{\frac{s'}{s'+1}} \Delta_n, & \text{if } \; s \le 3(1/p - 1/2), \end{cases}$$

where Δ_n is given by

$$\Delta_n = \exp\left\{\sqrt{\ln n}\,\sqrt{\ln \ln n}\left[\frac{A_1}{A_2}\left(3\nu + \frac{1}{A_2 \nu}\right) + o(1)\right]\right\},$$

and A_1, A_2, A_3 are functions of s.

Δ_n increases **slower than any power of** n but **faster than any power of** $\log n$ as $n \to \infty$.

SUMMARY

1. **The uniform case**: Conditions I, I* and II hold.
The convergence rates in the discrete model are independent of the number M and the choice of sampling points $\underline{u}$ and coincide with the convergence rates in the continuous model.

2. **The regular case**: Condition I holds but Condition II does not hold. Condition I* holds or, possibly, $|g_m(u_*)| = 0$.
One can point out the sampling scheme which delivers the fastest convergence rates, namely, sampling entirely at "the best possible" point u^*.

The uniform, or a more general pseudo-uniform, sampling may lead to convergence rates which differ from the convergence rates in the continuous model and are lower than when sampling is carried out entirely at the "best possible" point u^*.

SUMMARY

3. **The irregular case**: Condition I does not hold.
The convergence rates in the discrete model depend on a sampling strategy and, in addition, one cannot design a sampling scheme which delivers the highest convergence rates.

Example: the box-car kernel.

1. Sampling at any one point is, by far, not the best possible choice.
2. The highest convergence rates occur in the the continuous model.
3. The best choice for the discrete model is uniform sampling with a large value of $M \asymp n^{1/3}$ which is impractical in many situations.

CONCLUSIONS

In the regular case, one should be extremely careful when replacing a discrete model by its continuous counterpart.

In the irregular case, deal with the model at hand (discrete or continuous) paying uttermost attention to sampling strategies

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