Some properties of variance-based sensitivity indices for spatially distributed models

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GDR MASCOT NUM
Villard de Lans - March, 23^{th}, 2011

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  - black box
  - deterministic
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- Properties of S_i if X_i or Y is spatially distributed ?

Model ACB-DE : assessment of flood damages

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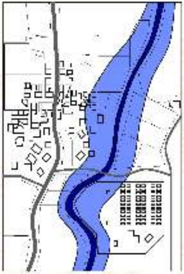
Hazard



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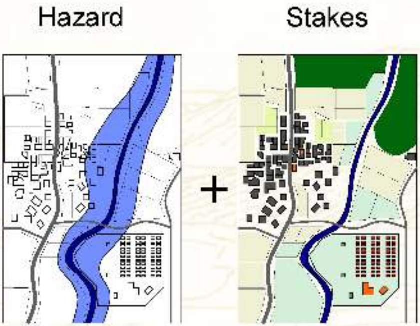

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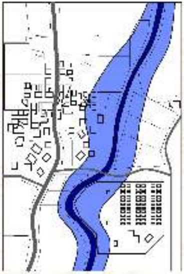

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\+

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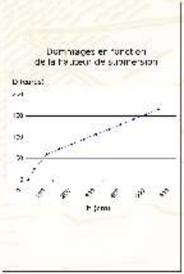

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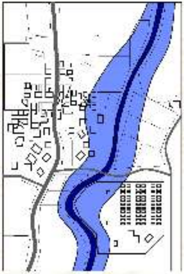
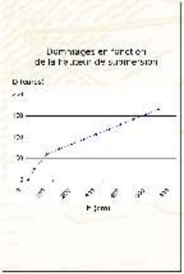
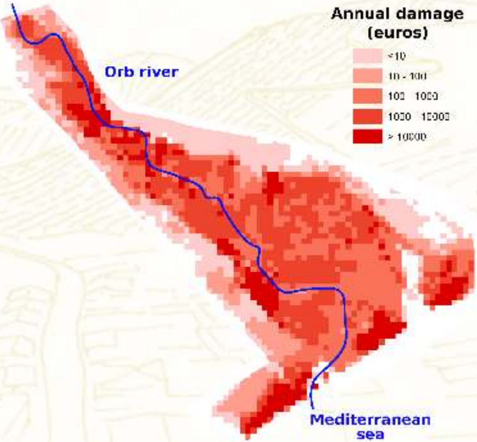
Hazard		Stakes		Vulnerability		Damage
	+		+		➤	1,3 M€

Model ACB-DE : inputs

	Nature	***Notation***
Water levels	Map	$Z_0(u) \in \mathbb{R}, \quad \forall u \in \mathbb{R}^2$
Landuse	Map	$Z_1(u) \in \mathbb{N}, \quad \forall u \in \mathbb{R}^2$
Terrain elevation	Map	$Z_2(u) \in \mathbb{R}, \quad \forall u \in \mathbb{R}^2$
Flood return period	Vector	$X_3 \in \mathbb{R}^5$
Infinite flood coefficient	Vector	$X_4 \in \mathbb{R}$
Damage curves	Vector	$X_5 \in \mathbb{R}^{100}$

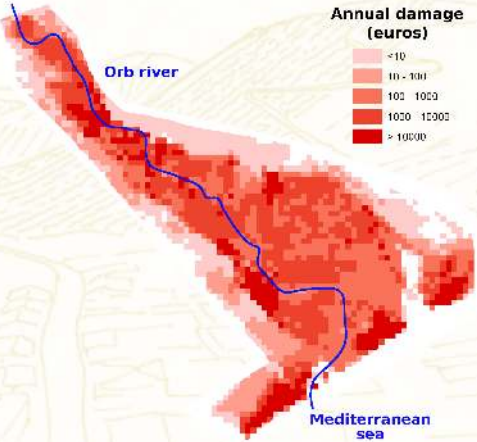
Model ACB-DE : output

Map of annual flood damages



Uncertainty of model output ?

Different questions for different stakeholders !



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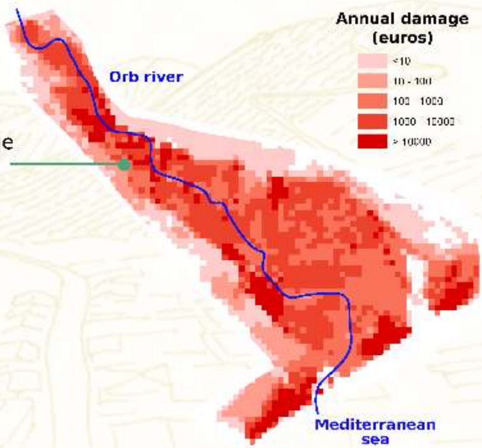

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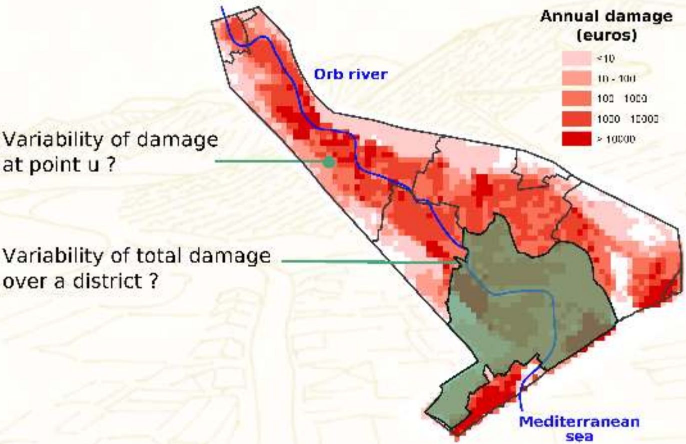


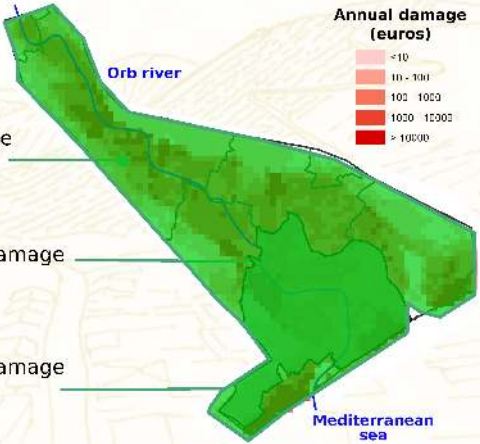
Uncertainty of model output ?

Different questions for different stakeholders !

Variability of damage at point u ?

Variability of total damage over a district ?

Variability of total damage over the territory ?



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→ Sensitivity indices of model inputs on various spatial support

Spatially distributed model $\mathcal{M}$

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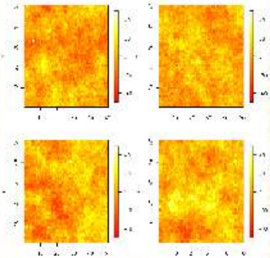
- $Z(u)$ Gaussian Random Field *(water levels)*
- $Y(u)$ a random field *(map of flood damage)* :

$$\forall u \in \mathcal{D}, \quad Y(u) = \psi\left[X, Z(u)\right]$$

with $\psi(.,.) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$

Model input : Gaussian random field $Z(u)$

- Gaussian Random Field
- Domain $\mathcal{D} \subset \mathbb{R}^2$
- Order 2 stationary



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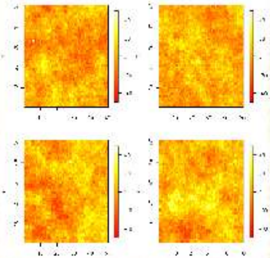

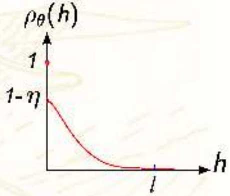

$$\forall u \in \mathcal{D}, \quad E\left[Z(u)\right] = \mu$$

$$\text{Cov}\left[\,Z(u), Z(u+h)\,\right] = \sigma_Z^2 \cdot \rho_\theta(h)$$

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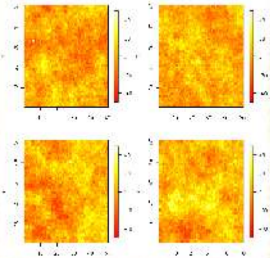

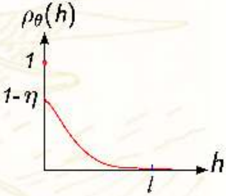

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Parameter $\theta = (\eta$: nugget, l : range$)$

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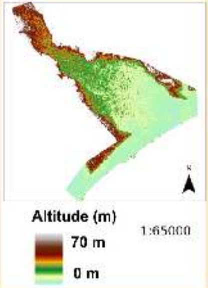
- Created from aerial photography
- Uncertainty : measure errors + interpolation errors



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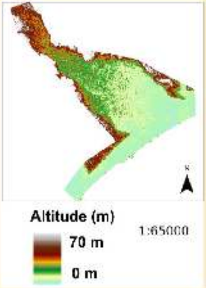
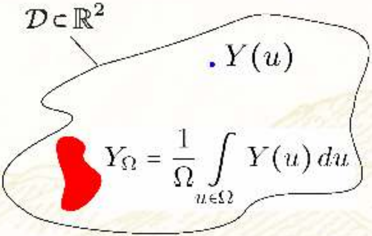


Modelling uncertainty on spatial model input

- 500 ground-validation points
- Estimation of mean error
- Estimation of covariance function

Model output : random field $Y(u)$



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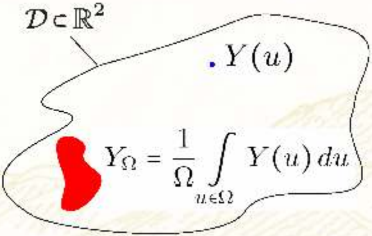

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$$\sigma_Y^2(u) = \mathrm{Var}\left[Y(u)\right]$$

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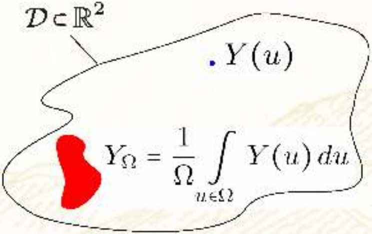

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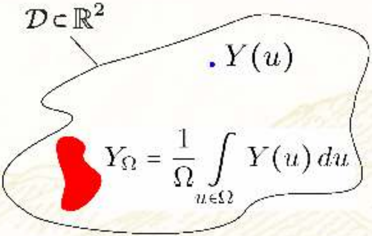

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- Block variance :

$$\sigma_Y^2(\Omega) = \mathrm{Var}\left[Y_\Omega\right]$$

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→ transformation of a Gaussian random field

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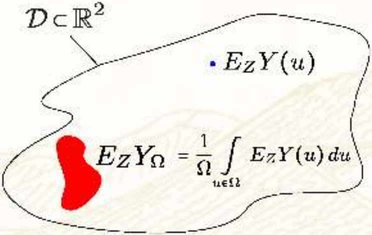

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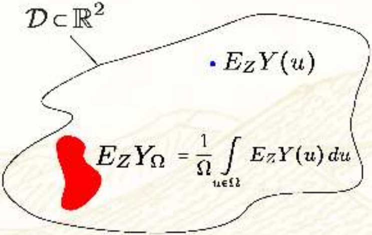

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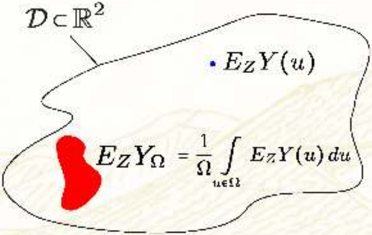

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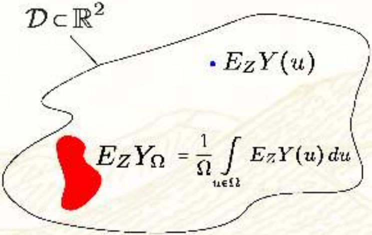

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$$\sigma^2_{E_Z Y}(\Omega) = \text{Var}\left[(E_Z Y)_\Omega\right] = \text{Var}\left[E_Z(Y_\Omega)\right]$$

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Point-based and zonal sensitivity indices

Property

Using previous notations :

$$\frac{S_Z(\Omega)}{S_X(\Omega)} = \frac{S_Z}{S_X} \cdot \frac{\sigma^2_{E_Z Y}(\Omega)}{\sigma^2_{E_Z Y}}$$

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- $\sigma^2_{E_Z Y}(\Omega)$ depends on the covariance function of RF $E_Z Y(u)$
- We have $E_Z Y = \bar{\psi}(Z)$
- Condition on $\bar{\psi}$:

$$\int_{-\infty}^{\infty} \bar{\psi}^2(z) \cdot n(z)\, dz < \infty$$

where $n(.)$ is the $\mathcal{N}(0,1)$ pdf.

Hermite polynomials

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Consider sequence $(\chi_k)_{k\in\mathbb{N}}$ such that :

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The sequence $(\chi_k)_{k\in\mathbb{N}}$ forms an orthonormal basis of Hilbert space $L^2(\mathcal{N})$:

$$L^2(\mathcal{N}) = \left\{ f : \mathbb{R} \to \mathbb{R} \quad tq. \int_{-\infty}^{\infty} f^2(z) \cdot n(z)\, dz < \infty \right\}$$

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$$\bar{\psi} = \sum_{k=0}^{\infty} \lambda_k \cdot \chi_k \qquad \text{with } \lambda_k = E\left[\chi_k \cdot \bar{\psi}\right]$$

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$\bar{\psi} \in L^2(\mathcal{N})$ *and* $E_Z Y = \bar{\psi}(Z)$, *thus :*

$$\bar{\psi} = \sum_{k=0}^{\infty} \lambda_k \cdot \chi_k \qquad \text{with } \lambda_k = E\left[\chi_k \cdot \bar{\psi}\right]$$

$$E_Z Y(u) = \sum_{k=0}^{\infty} \lambda_k \cdot \chi_k \left[Z(u)\right] \qquad \forall u \in \mathcal{D}$$

Properties of random field $E_Z Y(u)$

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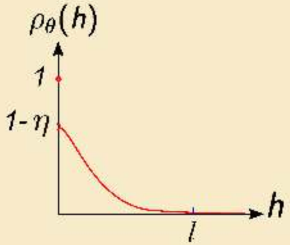
Property (Covariance function)

$$\forall u \in \mathcal{D}, \quad Cov\left[E_Z Y(u), E_Z Y(u+h)\right] = \sum_{k=0}^{\infty} \lambda_k^2 \cdot \sigma_Z^{2k} \cdot \rho_0^k(h)$$

Influence of range l

Property

- Assume : $\forall h > 0, \quad \frac{\partial \rho_\theta}{\partial l}(h) > 0$



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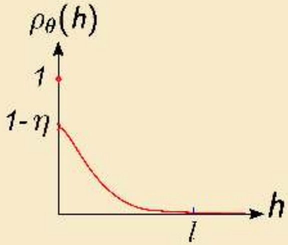

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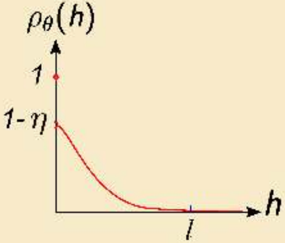

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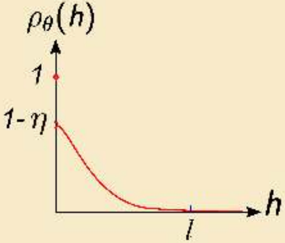
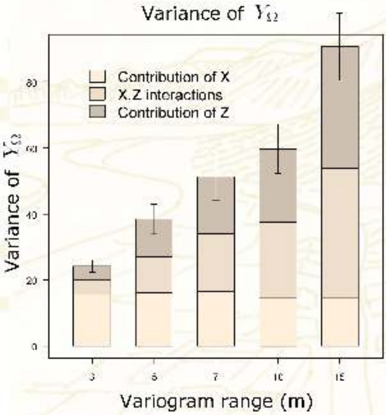
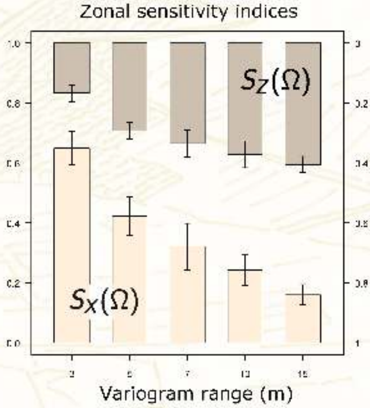


- Short range $l \rightarrow$ averaging of local errors $\rightarrow$ low $S_Z(\Omega)$
- Long range $l \rightarrow$ no averaging of local errors $\rightarrow$ high $S_Z(\Omega)$

Influence of range l - numerical example

- exponential covariance $\rho(h)$
- support $\Omega = \mathcal{D}$





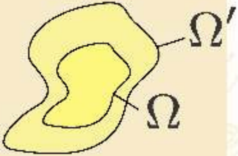
TETIS

Change of support

Change of support

Property

- Let $\Omega \subset \mathcal{D}$ and Ω' homothetic transformation of Ω of center O and ratio $\tau > 1$.



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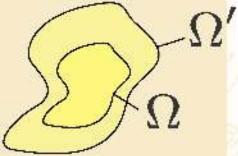

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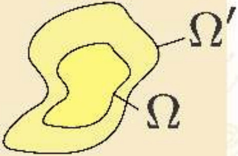

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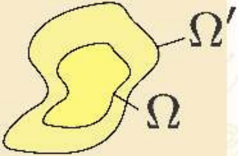
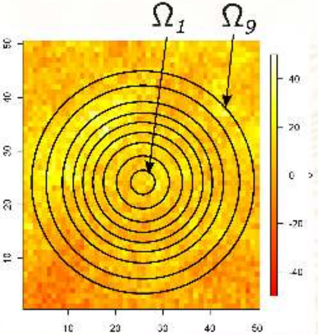


Larger zone $\Omega \rightarrow$ averaging of local errors $\rightarrow$ lower $S_Z(\Omega)$

Change of support - numerical example



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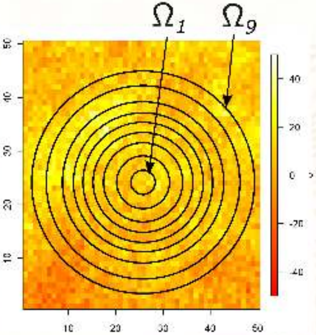

- Exponential covariance $\rho(h)$

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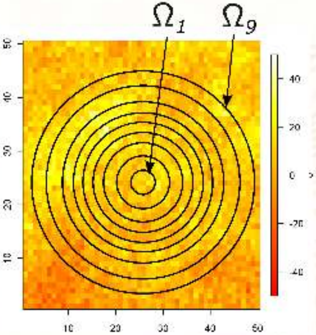

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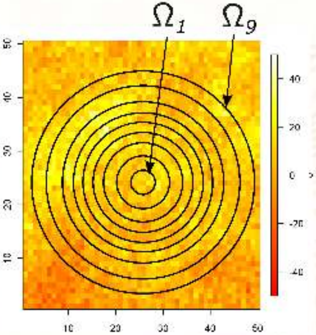
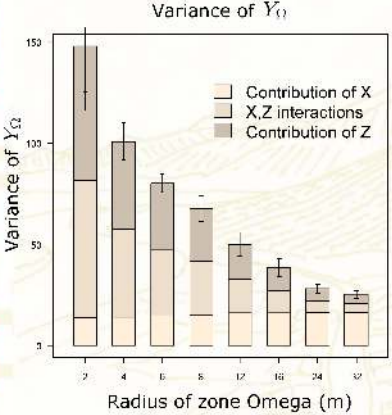
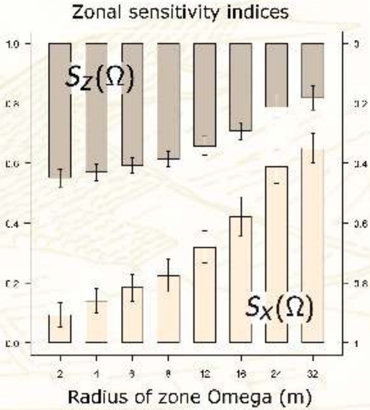


- Exponential covariance $\rho(h)$
- Zones Ω_1 to Ω_9 of increasing size
- Outputs of interest Y_{Ω_1} to Y_{Ω_9} :

$$Y_{\Omega_i} = \frac{1}{\Omega_i} \int_{u \in \Omega_i} Y(u)\, du$$

Change of support - numerical example





Approximation of zonal indices on a grid

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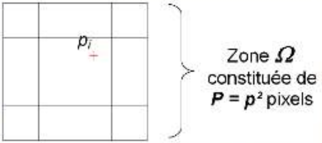
- Zone Ω represented on a linear grid



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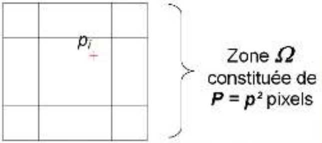

- Approximation of output of interest : $\tilde{Y}_\Omega = \frac{1}{P}\sum_{i=1}^{P} Y(p_i)$

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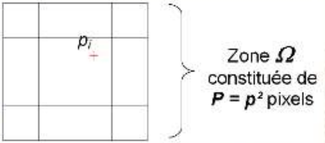

- Approximation of output of interest : $\tilde{Y}_\Omega = \frac{1}{P} \sum_{i=1}^{P} Y(p_i)$
- Approximation of zonal sensitivity indices : $\tilde{S}_X^\Omega = S_X(\tilde{Y}_\Omega)$
- Bias ε (for ψ linear) :

$$\varepsilon \underset{p \to \infty}{\sim} \frac{K}{p}$$

Other properties (ongoing work)

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  - properties of $S_X(u)$ and $S_Z(u)$?
- Estimation of zonal sensitivity indices
  - various techniques for spatial inputs
  - sampling random field $Z(u)$?

1. Motivations and notations
2. Point-based and zonal sensitivity indices
3. Some properties of zonal sensitivity indices
4. Conclusion

First answers to the modeler

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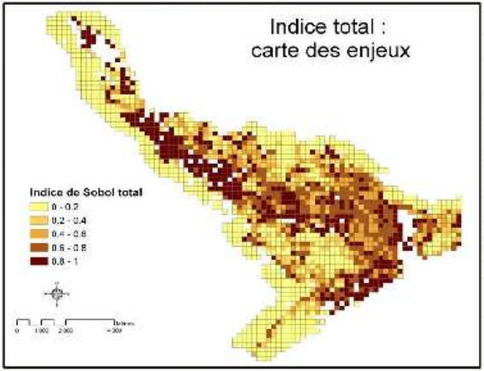
1. Maps of point-based sensitivity indices



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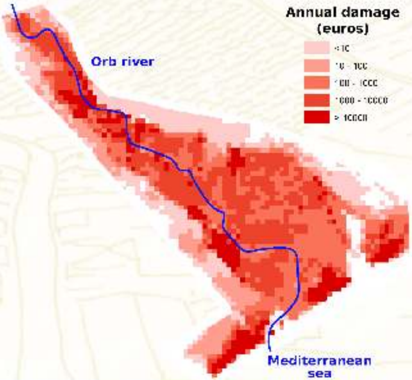
1. Maps of point-based sensitivity indices
2. Zonal sensitivity indices depend on the chosen spatial support Ω



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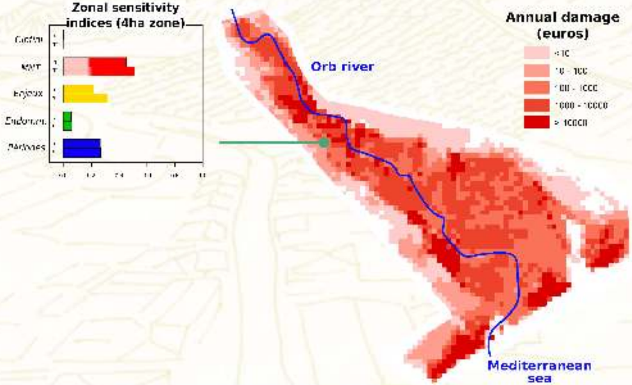

First answers to the modeler

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   - Individual stake : most important model input = water level
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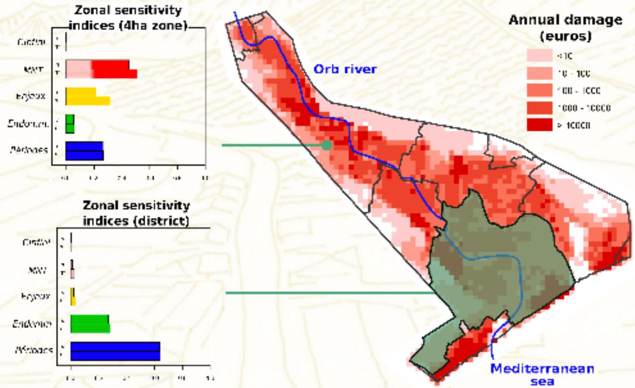

Variance-based spatial sensitivity analysis ?

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Variance-based spatial sensitivity analysis ?

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3. Limits
   - High CPU cost
   - « *Point-based* » model only

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Grelot, F. ; Bailly, J.-S. ; Blanc, C. ; Erdlenbruch, K. ; Meriaux, P. ; Saint-Geours, N. ; Tourment, R.
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Analyse de sensibilité de Sobol d'un modèle spatialisé pour l'évaluation économique du risque d'inondation
Journal de la Société Française de Statistique, 2011, 152, 24-46.

Saint-Geours, N. ; Lavergne, C. ; Bailly, J.-S. ; Grelot, F.
Is there room to optimise the use of geostatistical simulations for sensitivity analysis of spatially distributed models ?
Accuracy2010, Leicester, UK 20-23 July 2010

Merci pour votre attention

Indices de sensibilité (Sobol) (1/5)

Soit le modèle $Y = f(X_1, \ldots, X_p)$ avec $X_i \in \mathbb{R}$ et $X_i \perp X_j$.

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Si f est de carré intégrable, elle peut se décomposer (Hoeffding, 1948) en :

$$f(x_1, \ldots, x_p) = f_0 + \sum_{i=1}^{p} f_i(x_i) + \sum_{1 \leq i < j \leq p} f_{ij}(x_i, x_j) + \ldots + f_{1 \ldots p}(x_1, \ldots, x_p)$$

Indices de sensibilité (Sobol) (2/5)

La décomposition est unique sous la condition :

$$\forall u \subset \{1, \ldots, p\}, \quad \forall i \in u, \quad E_{X_i}[\, f_u \,] = 0$$

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On a alors $f_u \perp f_v$ et :

$$\begin{aligned} f_0 &= E[Y] \\ f_i(X_i) &= E[Y|X_i] - f_0 \\ f_{ij}(X_i, X_j) &= E[Y|X_i, X_j] - f_i - f_j - f_0 \end{aligned} \qquad (1)$$

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La variance de Y se décompose alors en :

$$V(Y) = \sum_{i=1}^{p} V(f_i) + \sum_{1 \leq i < j \leq p} V(f_{ij}) + \ldots + V(f_{1 \ldots p}) \tag{2}$$

Indices de sensibilité (Sobol) (3/5)

Pour $u \subset \{1, \ldots, p\}$, on définit

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$$S_u = \frac{\text{var}(f_u)}{\text{var}(Y)} \quad \forall u \subset \{1, \ldots, p\}$$

Indices de sensibilité (Sobol) (4/5)

Pour un modèle à p variables d'entrées indépendantes :

$$1 = S_1 + \cdots + S_d + \cdots + S_{1,\cdots,d}$$

Indices de sensibilité (Sobol) (4/5)

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plus l'indice est grand, plus la variable ou le groupe de variables est important vis à vis de la variance de Y

Indices de sensibilité (Sobol) (5/5)

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Indice de sensibilité de premier ordre

$$S_i = \frac{V(f_i)}{V(Y)} = \frac{\mathrm{Var}\left[E\left(Y \mid X_i\right)\right]}{\mathrm{Var}(Y)}$$

$\Rightarrow$ réduction espérée de la variance si l'on fixe X_1

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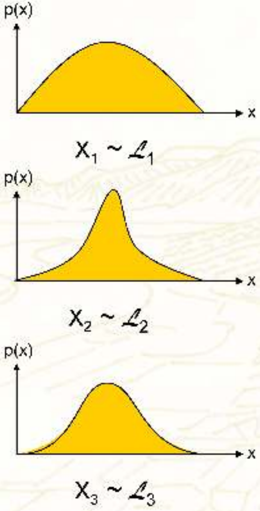
Indice de sensibilité total

$$ST_i = S_i + S_{i,1} + S_{i,2} + \cdots + S_{1,\ldots,d}$$

$\Rightarrow$ variance résiduelle de Y lorsque tous les X_j sauf X_i sont fixés

Indices de sensibilité : intuition



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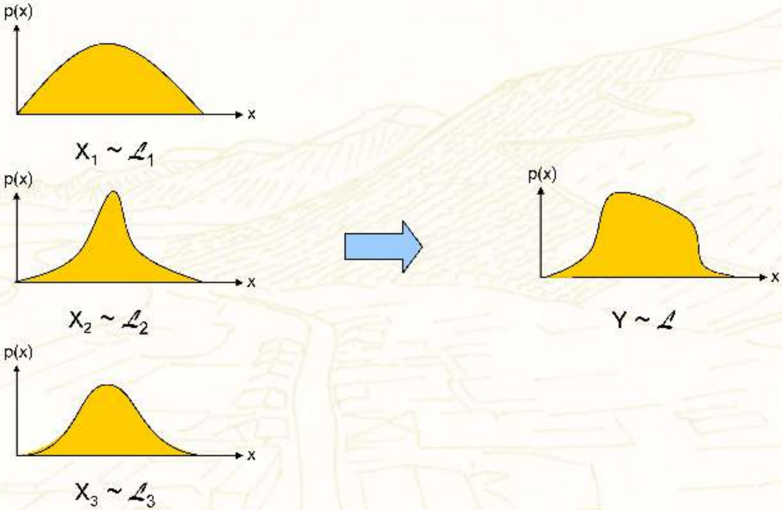

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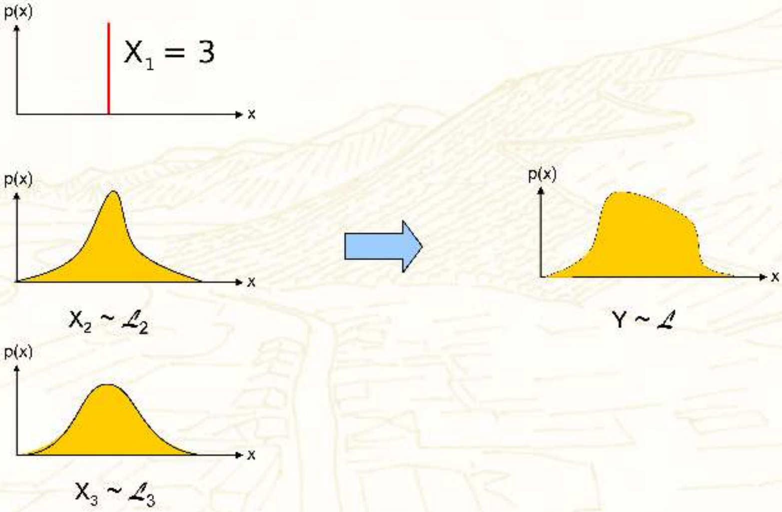

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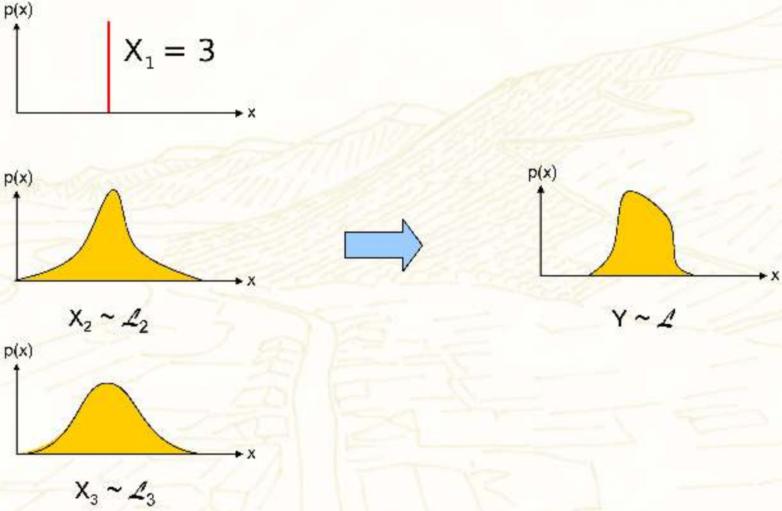

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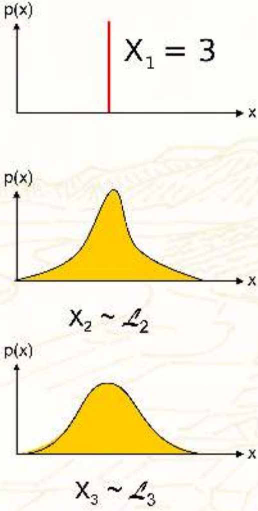
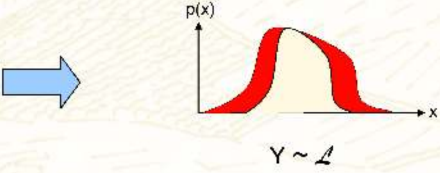
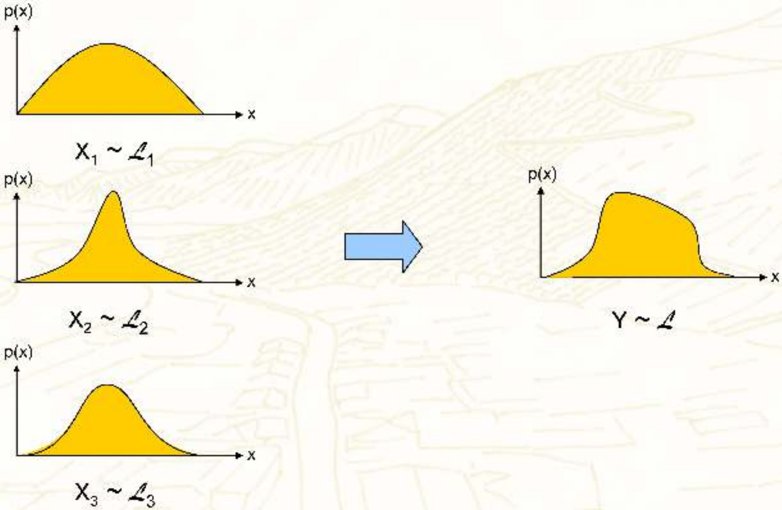


Indice de sensibilité
de premier ordre S_1



Réduction espérée de la variance de Y lorsque X1 est fixé

Indices de sensibilité : intuition



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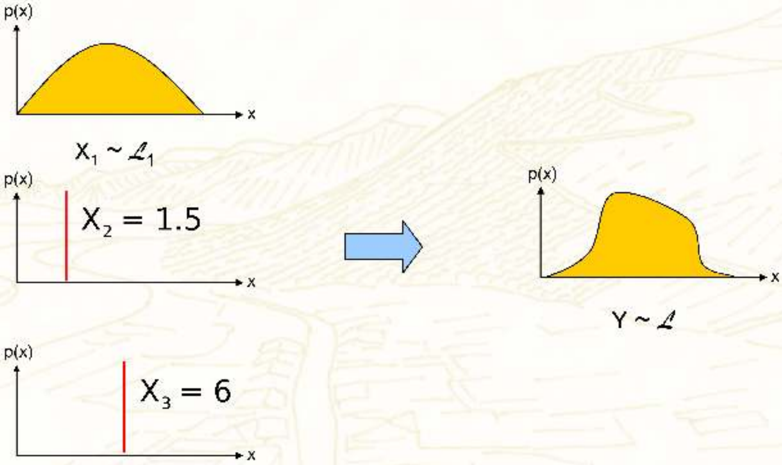

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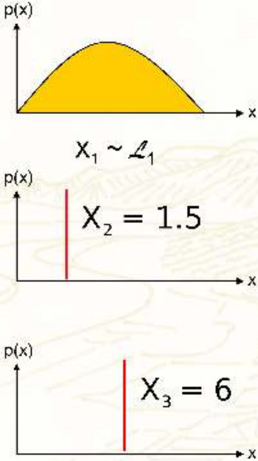
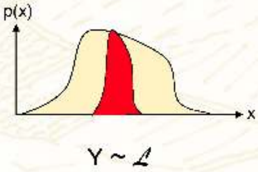


Indice de sensibilité
total ST_1



Variance résiduelle de Y lorsque X2 et X3 sont fixés

Indices de sensibilité : estimation

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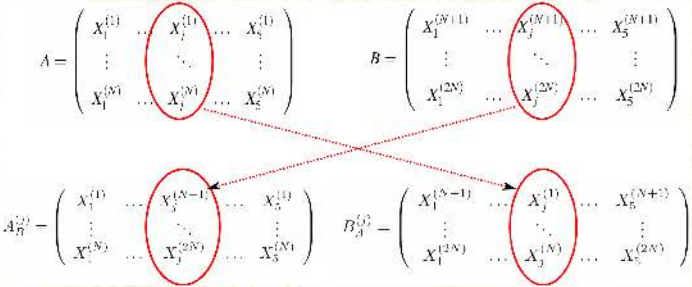
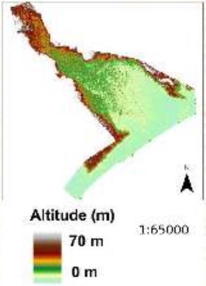
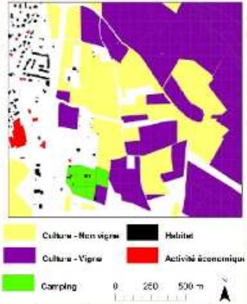
- estimation de type Monte-Carlo proposée par Sobol (1993)
- taille de l'échantillon nécessaire : $N \sim 1000 \cdot p$



Estimation de l'indice de sensibilité S_Z

- Rappel : indices de sensibilité estimés par une procédure de type Monte-Carlo (Sobol)





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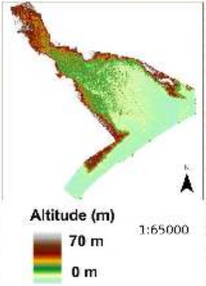
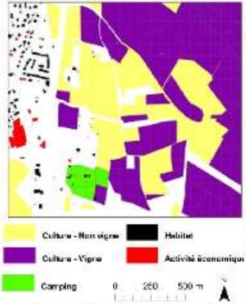
- Approche 1 : considérer Z comme un groupe de variables





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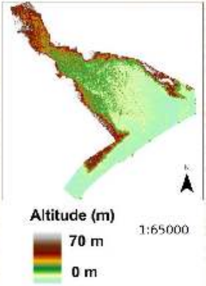

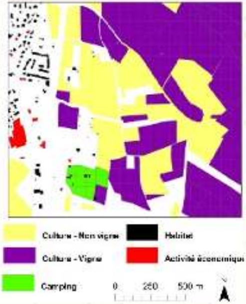

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Indices de sensibilité généralisés (Lamboni, 2010)

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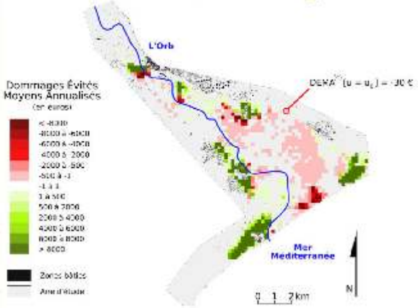
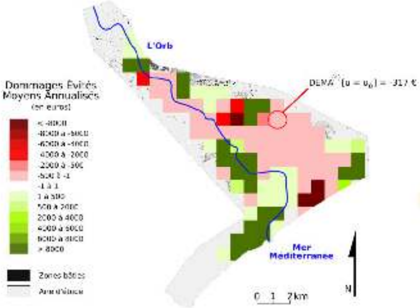
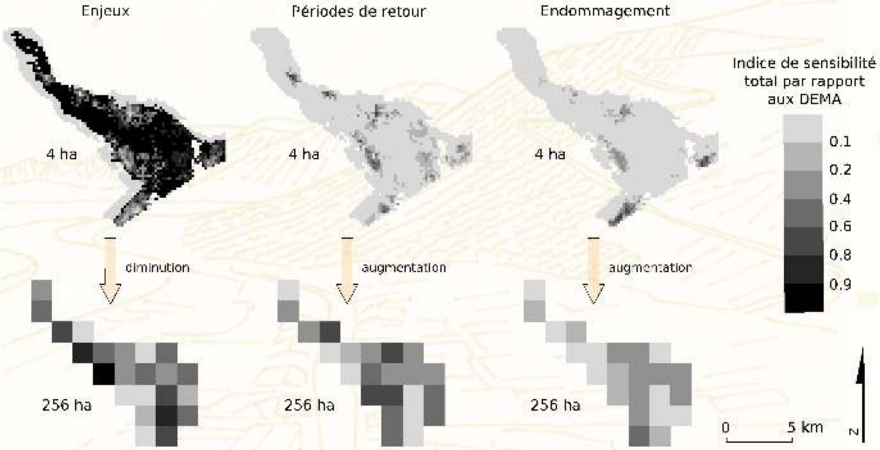
  $\rightarrow$ faible influence de Z

Cartes des Dommages Evités à différentes échelles

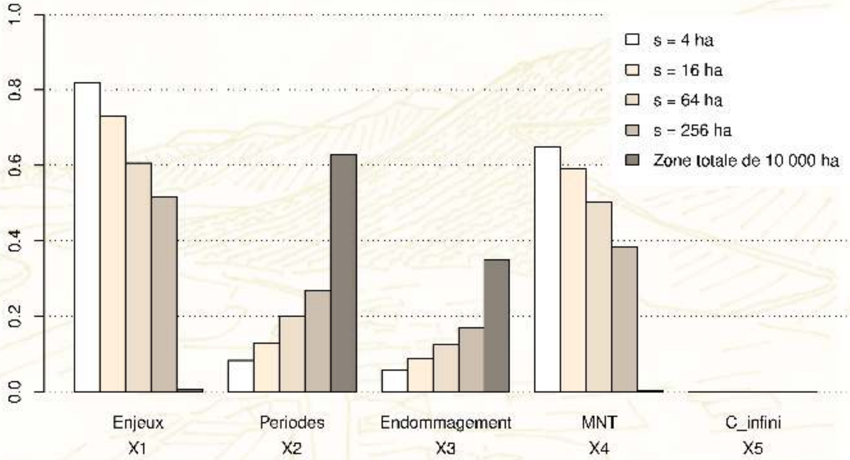




Cartes d'indices de Sobol à différentes échelles

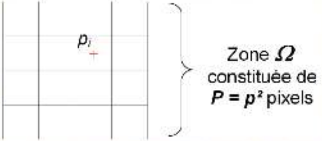


Indices de sensibilité moyens



Influence de la discrétisation du champ Z

- Discrétisation de la zone Ω supposée carrée



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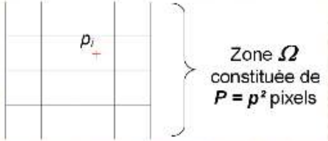

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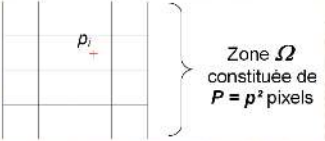

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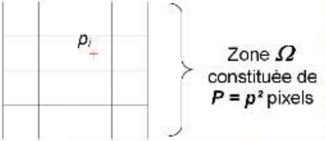

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Influence de la discrétisation du champ Z

Ecart entre indices approchés et valeurs exactes

- $\epsilon = \tilde{S}_X^\Omega - S_X^\Omega$
- ϵ linéaire en : $\frac{\eta\sigma_Z^2}{P} + \left[\bar{\gamma}(\nu,\nu) - \frac{1}{P^2}\sum_{i,j=1}^{P}\gamma^{-}(d_{p_i,p_j})\right]$
- $\lim_{p\to\infty} \epsilon = 0$, vitesse de convergence en $\frac{1}{p}$