Global sensitivity analysis for models with correlated input parameters

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Modern engineering makes a massive use of numerical simulations (CAE, FE-models) for the robust design of complex systems.

The structure can be decomposed into submodels representing a component, a scale of modelling or a physical discipline.

The computational workflow consists in an imbrication of submodels.
Robust design optimization

*Uncertainties* that may affect the system response, such as environmental loading, material properties or manufacturing tolerances, are taken into account.

Uncertainty quantification

Parameters of the system are modelled by a *random vector* $X \sim f_X(x)$.

The corresponding parameter of interest $Y = M(X)$ is studied.

*Sensitivity analysis* aims at identifying the important design variables through sensitivity measures:

- *Importance factors* in reliability analysis (FORM/SORM),
- *Sobol’ indices* are derived from the decomposition of the variance of the system output.
Issues of nested modelling

Of interest is often the sensitivity of the last level parameter, or final performance, to output parameters from the previous levels.

The joint distribution of these parameters, margins + copula (correlation), is implicitly defined by uncertainty propagation.

There is a need to develop sensitivity analysis tools for complex computational workflows.
Outline

1. Tools for global sensitivity analysis
2. Methods for dependent variables
3. Application
Outline

1. Tools for global sensitivity analysis
   - Sensitivity indices
   - Polynomial chaos expansion

2. Methods for dependent variables

3. Application
The ANOVA decomposition

Let us first consider an independent input random vector $X$ of dimension $n$ and a variable of interest $Y$ defined by $Y = M(X)$. The model $M$ can be uniquely decomposed by:

$$Y = M(X)$$

$$= M_0 + \sum_{i=1}^{n} M_i(X_i) + \sum_{1 \leq i < j \leq n} M_{i,j}(X_i, X_j) + \ldots + M_{1,\ldots,n}(X_1, \ldots, X_n)$$

$$= \sum_{u \subseteq \{1, \ldots, n\}} M_u(X_u)$$

where the summands have zero mean, are orthogonal and where the output variance can be written as:

$$\nabla [Y] = \sum_{u \subseteq \{1, \ldots, n\}} \nabla [M_u(X_u)]$$

Sobol’ first order and total indices for the subset of variables $X_u$ are defined by:

$$S_{1u} = \frac{\nabla [\mathbb{E} [Y | X_u]]}{\nabla [Y]}$$

and

$$S_{Tu} = 1 - \frac{\nabla [\mathbb{E} [Y | X_{\bar{u}}]]}{\nabla [Y]}$$
Let us denote by $Y^u = \mathcal{M}(X_u, X'_u)$ where $X'_u$ is an independent copy of $X_u$. The first order Sobol’ indices can also be written:

$$S_{1u} = \frac{\text{Cov}[Y, Y^u]}{\sqrt{\text{Var}[Y]}}$$

We now consider $N$-samples $X$, $Y$ and $Y^u$ of $X$, $Y$ and $Y^u$. Sobol’ first order indices can be estimated by:

$$S_{1u}^N = \frac{\frac{1}{N} \sum y_i y_i^u - \left(\frac{1}{N} \sum y_i\right) \left(\frac{1}{N} \sum y_i^u\right)}{\frac{1}{N} \sum y_i^2 - \left(\frac{1}{N} \sum y_i\right)^2}$$  \hspace{1cm} (1)

or more efficiently by:

$$T_{1u}^N = \frac{\frac{1}{N} \sum y_i y_i^u - \left(\frac{1}{N} \sum \left[\frac{y_i + y_i^u}{2}\right]\right)^2}{\frac{1}{N} \sum \left[\frac{y_i^2 + y_i^u}{2}\right] - \left(\frac{1}{N} \sum \left[\frac{y_i + y_i^u}{2}\right]\right)^2}$$  \hspace{1cm} (2)
The Fisher transform:

The Fisher transformation of a correlation coefficient from a two Gaussian random variables $N$-sample $X$ is given by:

$$z = \frac{1}{2} \ln \frac{1 + \hat{\rho}}{1 - \hat{\rho}} = \text{artanh}(\hat{\rho}), \quad \text{with} \quad \hat{\rho} = \frac{\text{Cov}[X_1, X_2]}{\sigma_1 \sigma_2}$$

and $z$ as the following asymptotic behaviour:

$$z \sim \mathcal{N} \left( \frac{1}{2} \ln \frac{1 + \rho}{1 - \rho}, \frac{1}{\sqrt{N - 3}} \right)$$

By considering the sensitivity index $S_{1u}$ as a linear correlation coefficient $\rho \left( \mathcal{M}(X_u, X_{\bar{u}}), \mathcal{M}(X_u, X'_{\bar{u}}) \right)$, a $\alpha$-confidence interval is given by:

$$\left[ \text{tanh} \left( \frac{1}{2} \ln \frac{1 + \hat{S}_u}{1 - \hat{S}_u} + \Phi^{-1} \left( \frac{\alpha}{2} \right) \right), \text{tanh} \left( \frac{1}{2} \ln \frac{1 + \hat{S}_u}{1 - \hat{S}_u} + \Phi^{-1} \left( 1 - \frac{\alpha}{2} \right) \right) \right]$$

[Martinez (2011)]
Polynomial chaos expansion

**Limitation:**
Although estimators are efficient, they still require a large number ($N = 10^3 – 10^4$) of model evaluations for an accurate estimation of the indices → This is hardly achievable when a coupling procedure with a FE-model is performed.

**Polynomial chaos expansion:**

$$Y = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} \Psi_\alpha(X)$$

where:

- $\Psi_\alpha$ Polynomial chaos basis, e.g. $\Psi_\alpha(X) = \prod_{i=1}^{n} He_{\alpha_i}(\xi_i)$ (multivariate Hermite polynomials) for Gaussian variables
- $a_{\alpha}$ Coefficients to be determined
Polynomial chaos expansion

**In practice**: One only retains the polynomials $\Psi_\alpha$ whose total degree is less than $p$:

$$Y \approx \sum_{|\alpha| \leq p} a_\alpha \Psi_\alpha(X)$$

**Computation of the coefficients by regression**:

$$Y = \sum_{j=0}^{P-1} a_j \Psi_j(X) + \epsilon_P \equiv a^T \Psi(X) + \epsilon_P, \quad P = \binom{n+p}{p}$$

Can be solved by least square regression:

$$\hat{a} = \arg\min \frac{1}{N} \sum_{i=1}^{N} \left( M(x^{(i)}) - a^T \Psi(x^{(i)}) \right)^2, \quad a \in \mathbb{R}^P$$

$Y$ is characterized by the coefficients $\hat{a} = \{a_0, \ldots, a_{P-1}\}^T$ of the expansion.
PCE-based Sobol’’ indices (1)

- **Polynomial chaos approximation**:

\[ \mathcal{M}(X) \approx \mathcal{M}^{PC}(X) = \sum_{\|\alpha\| \leq p} a_{\alpha} \Psi_{\alpha}(X) \]  

- **Multi-indices notation**:

\[ I_{i_1, \ldots, i_s} \text{ such as only the indices } (i_1, \ldots, i_s) \text{ are non zero:} \]

\[ I_{i_1, \ldots, i_s} = \left\{ \alpha : \begin{array}{l} \alpha_k > 0 \quad \forall k = 1, \ldots, M \quad k \in (i_1, \ldots, i_s) \\ \alpha_k = 0 \quad \forall k = 1, \ldots, M \quad k \notin (i_1, \ldots, i_s) \end{array} \right\} \]  

Summands can be obtained by gathering functions of the subset of variables \( I_{i_1, \ldots, i_s} \):

\[ \mathcal{M}^{PC}(X) = a_0 + \sum_{i=1}^{n} \sum_{\alpha \in I_i} a_{\alpha} \Psi_{\alpha}(X_i) + \ldots + \sum_{\alpha \in I_1, 2, \ldots, n} a_{\alpha} \Psi_{\alpha}(X) \]
PCE-based Sobol’ indices (2)

Summands of the expansion $\mathcal{M}^{PC}(X)$ can be compared to those from the Sobol’ decomposition:

$$\mathcal{M}_{i_1,\ldots,i_s}(x_{i_1}, \ldots, x_{i_s}) = \sum_{\alpha \in I_{i_1,\ldots,i_s}} a_\alpha \Psi_\alpha (x_{i_1}, \ldots, x_{i_s}) \quad (6)$$

PCE-based Sobol’ first order and total indices are given by:

$$SU_{i_1,\ldots,i_s} = \frac{1}{\sigma_Y^{2,PC}} \sum_{\alpha \in I_{i_1,\ldots,i_s}} a_\alpha^2 = \frac{\sum_{\alpha \in I_{i_1,\ldots,i_s}} a_\alpha^2}{\sum_{0<|\alpha|\leq p} a_\alpha^2} \quad (7)$$

and

$$SU^{T}_{j_1,\ldots,j_t} = \sum_{(i_1,\ldots,i_s) \subset (j_1,\ldots,j_t)} SU_{i_1,\ldots,i_s} \quad (8)$$

Sobol’ indices are given by the coefficients of the expansion without any extra numerically expensive simulations.
Tools for global sensitivity analysis
Methods for dependent variables
Application

A moment-independent measure
Covariance decomposition of the model output
Observations
The $\delta$-sensitivity measure

**Principle**:

If the model output $Y = M(X)$ is affected by a input variable $X_i$, then the conditional distribution $f_{Y|X_i}(y)$ significantly differs from $f_Y(y)$.

**The $\delta$-sensitivity measure**:

Let us define the *shift* (blue area left) between the two distributions by:

$$s(X_i) = \int_{D_Y} \left| f_Y(y) - f_{Y|X_i}(y) \right| dy.$$

We define by $\delta_i$ the following index:

$$\delta_i = \frac{1}{2} \mathbb{E} \left[ s(X_i) \right]$$

$$= \frac{1}{2} \int_{D_{X_i}} \left[ \int_{D_Y} \left| f_Y(y) - f_{Y|X_i}(y) \right| dy \right] f_{X_i}(X_i) dx_i$$
Computational aspects

- **Kernel smoothing estimation** for the distributions:
  
  Approximation of the PDF using a $N$-sample:
  
  $$ \hat{f}_Y(y) = \frac{1}{h} \sum_{i=1}^{N} K \left( \frac{y - y(i)}{h} \right), \quad K(t) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} t^2 \right) $$

  where $K$ is the Gaussian kernel
  and $h = \left( \frac{4 \hat{\sigma}}{3N} \right)^{\frac{1}{5}}$ its optimal bandwidth.

- **Double quadrature loop** for the integrals:
  1. $s(X_i) = \int_{D_Y} \left| f_Y(y) - f_{Y|X_i}(y) \right| dy \approx \sum_{q_1=1}^{n_Q} \omega_{q_1} \left| f_Y(y_{q_1}) - f_{Y|X_i}(y_{q_1}) \right|$
  2. $\mathbb{E}[s(X_i)] = \int_{D_{X_i}} s(X_i)f_{X_i}(x_i)dx_i \approx \sum_{q_2=1}^{n_Q} \omega_{q_2} s(x_{i,q_2})$
A moment-independent measure
Covariance decomposition of the model output
Observations

**Δ-sensitivity measure based on the CDFs**  
[Borgonovo (2011)]

Δ-measures require large samples \( (N = 10^5) \) for an accurate estimation of the densities.

**Alternative computation scheme** :

\[
s(X_i) = 2 \Pr_Y \left( F_Y(y) > F_{Y|X_i}(y) \right) \\
- 2 \Pr_Y \left( F_Y(y) < F_{Y|X_i}(y) \right)
\]

That is :

\[
\delta_i = \mathbb{E} \left[ F_Y(y_1) - F_{Y|X_i}(y_2) \right. \\
\left. - F_Y(y_2) - F_{Y|X_i}(y_1) \right]
\]

where \( y_1, y_2 \) are the solutions of :

\[
f_Y(y) - f_{Y|X_i}(y) = 0, \quad y_1 < y_2.
\]
The *Ishigami function* is defined by:

\[ Y = \sin(X_1) + a \sin(X_2)^2 + b X_3^4 \sin(X_1) \]

with \( a = 7 \), \( b = 0.1 \) and \( X_i \sim U[-\pi, \pi], \ i = 1, 2, 3 \). We use \( N = 10^5 \) because of the shape of the output PDF.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( \delta_i )</th>
<th>( S_{1i} )</th>
<th>( S_{Ti} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>0.23</td>
<td>0.40</td>
<td>0.71</td>
</tr>
<tr>
<td>( X_2 )</td>
<td>0.37</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>0.10</td>
<td>0.00</td>
<td>0.31</td>
</tr>
<tr>
<td>Total</td>
<td>0.70</td>
<td>0.69</td>
<td>1.31</td>
</tr>
</tbody>
</table>

\( x_1 = 0.328, \ s(x_1) = 0.506 \)

\( x_2 = -1.353, \ s(x_2) = 0.826 \)

\( x_3 = 0.792, \ s(x_3) = 0.406 \)
**The ANCOVA decomposition**

**ANCOVA** stands for **ANalysis of COVAriance**.

By definition:

\[ \forall [Y] = \text{Cov} [Y, Y] \]

\[ = \text{Cov} \left[ Y, \sum_{i=1}^{n} M_i(X_i) + \sum_{1 \leq i < j \leq n} M_{i,j}(X_i, X_j) + \ldots + M_{1,\ldots,n}(X) \right] \]

\[ = \text{Cov} \left[ Y, \sum_{u \subseteq \{1,\ldots,n\}} M_u(X_u) \right] \]

It requires a functional decomposition of the model. Fortunately, one is given by the *polynomial chaos expansion*!

\[ Y = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \Psi_\alpha(X) \]

with:

\[ \sum_{\alpha \in u} a_\alpha \Psi_\alpha(X) \equiv M_u(X_u) \]
Sensitivity indices for dependent variables

Given:

\[ \nabla [Y] = \text{Cov} \left[ Y, \sum_{u \subseteq \{1, \ldots, n\}} M_u(X_u) \right] \]

We define a \textit{triplet of sensitivity indices}:

\[ S_u = \frac{\text{Cov} [Y, M_u(X_u)]}{\nabla [Y]} \approx \frac{\sum_{i=1}^{N} (Y(i) - \bar{Y})(M_u(x_u^{(i)}))}{\sum_{i=1}^{N} (Y(i) - \bar{Y})^2} \]

\[ S^S_u = \frac{\nabla [M_u(X_u)]}{\nabla [Y]} \approx \frac{\sum_{i=1}^{N} (M_u(x_u^{(i)}))^2}{\sum_{i=1}^{N} (Y(i) - \bar{Y})^2} \]

\[ S^C_u = \frac{\text{Cov} \left[ \sum_{v \subseteq \{1, \ldots, n\}, u \cap v = \{0\}} M_v(X_v), M_u(X_u) \right]}{\nabla [Y]} \approx \frac{\sum_{i=1}^{N} (\sum_v M_v(x_v^{(i)}))(M_u(x_u^{(i)}))}{\sum_{i=1}^{N} (Y(i) - \bar{Y})^2} \]

- \( S_u \): \textit{total} contribution of \( X_u \) to \( \nabla [Y] \)
- \( S^S_u \): \textit{structural} contribution of \( X_u \) \( \nabla [Y] \)
- \( S^C_u \): \textit{correlative} contribution of \( X_u \) \( \nabla [Y] \)

\[ S_u = S^S_u + S^C_u \]
A didactic example: model

Let us examine the following polynomial function:

\[ Y = M(X) = X_1 + X_2 + X_2^2 + X_1X_2 + 3 \]

with \( X_i \sim \mathcal{N}(0, 1), \ i = 1, 2. \)

Choosing \( p = 2 \), the *functional decomposition* reads:

\[ M(x) = M_0 + M_1(x_1) + M_2(x_2) + M_{1,2}(x_1, x_2) \]

with:

\[
\begin{align*}
M_0 &= 4 \times \Psi_{0,0}(x_1, x_2) = 4 \\
M_1(x_1) &= 1 \times \Psi_{1,0}(x_1, x_2) = x_1 \\
M_2(x_2) &= 1 \times \Psi_{0,1}(x_1, x_2) + \sqrt{2} \times \Psi_{0,2}(x_1, x_2) = x_2 + x_2^2 - 1 \\
M_{1,2}(x_1, x_2) &= 1 \times \Psi_{1,1}(x_1, x_2) = x_1x_2
\end{align*}
\]
A didactic example: independent variables

Let us first consider $X_1$, $X_2$ as independent variables, i.e. $F_X(x) = F_{X_1}(x_1)F_{X_2}(x_2)$. Sobol’ PCE-based indices are:

\[
S_{11} = 0.2, \quad ST_{1} = 0.4 \quad (S_{1,2} = 0.2) \\
S_{12} = 0.6, \quad ST_{2} = 0.8 \quad (S_{2,1} = 0.2)
\]

$X$ and $Y$ are $N$-samples used for the evaluation of the moments. Example with $N = 1000$:

\[
S_1 \approx \frac{\sum_{i=1}^{N} (x_1^{(i)})(y^{(i)} - \bar{y})}{\sum_{i=1}^{N} (y^{(i)} - \bar{y})^2}, \quad S_1^S \approx \frac{\sum_{i=1}^{N} (x_1^{(i)})^2}{\sum_{i=1}^{N} (y^{(i)} - \bar{y})^2}, \quad S_1^C \approx S_1 - S_1^S
\]

<table>
<thead>
<tr>
<th>$\rho_S$ = 0.2</th>
<th>$S_i$</th>
<th>$S_i^S$</th>
<th>$S_i^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.20</td>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td>$X_2$</td>
<td>0.60</td>
<td>0.60</td>
<td>0.00</td>
</tr>
<tr>
<td>$X_{1,2}$</td>
<td>0.20</td>
<td>0.20</td>
<td>0.00</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>1.00</td>
<td>1.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

\[\rightarrow\] Structural contributions are equal to the Sobol’ indices.  
\[\rightarrow\] Correlative contributions are zero.
A didactic example: correlated variables

Let us now consider $X_1, X_2$ as *correlated variables* with the following rank correlation matrix:

$$S = \begin{bmatrix} 1 & \rho_S \\ \rho_S & 1 \end{bmatrix}$$

We now have $F_X(x) = C_R(F_{X_1}(x_1), F_{X_2}(x_2))$, $R_{ij} = 2 \sin\left(\frac{\pi}{6} S_{ij}\right)$, where $C_R$ is the Gaussian copula of $(X_1, X_2)$.

<table>
<thead>
<tr>
<th>$\rho_S$ = 0.2</th>
<th>$S_i$</th>
<th>$S^S_i$</th>
<th>$S^C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.19</td>
<td>0.17</td>
<td>0.02</td>
</tr>
<tr>
<td>$X_2$</td>
<td>0.59</td>
<td>0.51</td>
<td>0.08</td>
</tr>
<tr>
<td>$X_{1,2}$</td>
<td>0.22</td>
<td>0.18</td>
<td>0.04</td>
</tr>
<tr>
<td>$\sum$</td>
<td>1.00</td>
<td>0.86</td>
<td>0.14</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$\rho_S$ = 0.8</th>
<th>$S_i$</th>
<th>$S^S_i$</th>
<th>$S^C_i$</th>
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</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>0.19</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>$X_2$</td>
<td>0.52</td>
<td>0.29</td>
<td>0.23</td>
</tr>
<tr>
<td>$X_{1,2}$</td>
<td>0.29</td>
<td>0.14</td>
<td>0.15</td>
</tr>
<tr>
<td>$\sum$</td>
<td>1.00</td>
<td>0.53</td>
<td>0.47</td>
</tr>
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*Correlative contributions increase with the correlation whereas structural contributions decrease.*
Pros and cons

$\delta$-measure

**Advantages** + :
- moment-free
- no hypothesis on the dependence structure

**Drawbacks** − :
- non-unity sum
- require accurate approximation of the conditional distributions
- PDF: 2-loop integration, CDF: 1-loop integration

ANCOVA decomposition

- consistent with Sobol’ indices ($I$)
- a triplet of indices ($S_i, S_i^S, S_i^C$)
- separate structural and correlative contributions

- require a functional decomposition of the model
- complex to interpret
- 3 (at least 2) indices to compute

We now have two alternative methods for global sensitivity analysis for models with correlated input parameters.
Outline

1. Tools for global sensitivity analysis
2. Methods for dependent variables
3. Application
Definition and probabilistic model:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unity</th>
<th>Distribution</th>
<th>µ</th>
<th>σ/µ</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>kN</td>
<td>Gumbel</td>
<td>100</td>
<td>10%</td>
</tr>
<tr>
<td>ρ</td>
<td>kg.m(^{-3})</td>
<td>Weibull</td>
<td>7860</td>
<td>10%</td>
</tr>
<tr>
<td>L</td>
<td>m</td>
<td>Normal</td>
<td>5</td>
<td>5%</td>
</tr>
<tr>
<td>ω(_{CD})</td>
<td>mm</td>
<td>Normal</td>
<td>125</td>
<td>10%</td>
</tr>
<tr>
<td>t</td>
<td>mm</td>
<td>Normal</td>
<td>250</td>
<td>10%</td>
</tr>
</tbody>
</table>
Tools for global sensitivity analysis
Methods for dependent variables
Application

Bracket Structure: model

- The output of interest is the bending stress \( \sigma_B \) reading:

\[
\sigma_B = \frac{6M_B}{\omega_{CD}t^2} \quad \text{with} \quad M_B = \frac{PL}{3} + \frac{\rho g \omega_{CD} t L^2}{18}
\]

- Computing parameters: \( p = 7 \) \( (Q^2 = 1.000) \), \( N = 10^4 \).

\[
\begin{array}{cccccc}
\text{Parameter} & S_i & S_i^S & S_i^C & \delta_i \\
P & 0.00 & 0.00 & 0.00 & 0.02 \\
\rho & 0.00 & 0.00 & 0.00 & 0.01 \\
L & 0.42 & 0.42 & 0.00 & 0.27 \\
\omega_{CD} & 0.02 & 0.00 & 0.02 & 0.02 \\
t & 0.56 & 0.54 & 0.02 & 0.33 \\
\end{array}
\]

\[
\sum = 1.00 \quad 0.96 \quad 0.04 \quad 0.65
\]
We now study the influence of $\rho_S(\omega_{CD}, t)$ on the structural and correlative contributions.

**Structural contributions**

**Correlative contributions**

In this case:
- Correlative contributions of $\omega_{CD}$ and $t$ strongly depends on $\rho_S$.
- They are negative when $\rho_S < 0$ and positive when $\rho_S > 0$. 

Yann Caniou (PHIMECA)
Conclusion

General remarks on GSA:

1. Sensitivity indices represent the contribution of one (or more) input parameter on the variance of the model output.
2. *Polynomial chaos expansions* are a powerful metamodelling tool to deal with time-demanding models.
3. Sobol’ indices can be directly computed from the coefficients of the expansion.

Correlated input parameters:

1. The nested modelling of complex structures involves correlation between the parameters.
2. Classical variance-based methods cannot be applied due to the hypothesis of independence.
3. A *moment-free method* has been presented to circumvent the issue of correlated input parameters.
4. The so-called *ANCOVA decomposition* represents a generalization of the ANOVA decomposition.
5. Polynomial chaos expansions offer a *functional decomposition* of the model.
*Adaptive surrogate models for reliability analysis and reliability-based design optimization.*  

*A new uncertainty importance measure.*  
Reliability Engineering and System Safety 92, 771–784.

*Moment Independent Importance Measures : New Results and Analytical Test Cases.*  
Risk Analysis 31, 404–428.

*Stochastic Finite Elements : A Spectral Approach (Revised edition).*  
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*Asymptotic normality and efficiency of two Sobol index estimators.*

*Global Sensitivity Analysis for Systems with Independent and/or Correlated Inputs.*  
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Analyse de sensibilité globale par décomposition de la variance.  
GdR Ondes & MASCOTNUM

*Sensitivity estimates for nonlinear mathematical models.*  
Mat Model 2, 112–8.

*Global sensitivity analysis using polynomial chaos expansions.*  

*Kernel smoothing.*  
Chapman and Hall.
Thank you for your attention.