Revisiting Morris method: 
A polynomial algebra for design definition with improved efficiency and observability

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Plan

1. Problem formulation and summary of contributions
2. Polynomial representation of subgraphs
3. Generation of \((d, m)\)-edge equitable subgraphs
4. Generation of \((d, c)\)-cycle equitable subgraphs: \(H^d_c\)
5. Size of designs
6. Example
7. Summary and further work
We’ll be looking at two related problems
Problem 1

Find subgraphs $G_m^d \subset Q_d$ of the $d$-dimensional hypercube with the property:

$\forall i \in \{1, \ldots, d\}$, the number of edges joining nodes that differ only in the $i$-th coordinate is equal to $m$.

We say that graphs with this property are $(d, m)$-edge equitable.

$\text{(3, 2)-edge equitable}$  $\text{Not (3, m)-edge equitable}$
Problem 2

Find edge equitable subgraphs $H^d_c \subset Q_d$ of the $d$-dimensional hypercube with the property:

$$\forall i \neq j \in \{1, \ldots, d\}, \text{ the number of cycles in coordinates } i, j \text{ is equal to } c.$$

We say that graphs with this property are $(d, c)$-cycle equitable.

\begin{align*}
\begin{array}{c|cccc}
(i, j) & 2 & 3 & 4 \\
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 \\
\end{array}
& \quad \text{(4, 1)-cycle equitable} \\
\begin{array}{c|cc}
(i, j) & 2 & 3 \\
1 & 1 & 0 \\
2 & 1 & 0 \\
\end{array}
& \quad \text{not cycle equitable}
\end{align*}
Motivation

Morris elementary effects screening method for sensitivity analysis (Technometrics, 1991)

Commonly used screening method for analysis of $f : \mathbb{R}^d \rightarrow \mathbb{R}$
- Partitions input factors into linear, negligible and non-linear/mixed
- Makes no assumptions about $f$
- Simple (linear in the number of inputs), OAT global method.

Based on statistical analysis of

Elementary effect along direction $i \in \{q, \ldots, d\}$

$$d_i(y) \triangleq \frac{1}{\Delta} [f(y + \Delta e_i) - f(y)], \quad i \in \{1, \ldots, d\}$$
Standard Morris method

OAT method:

A complete set of $d$ elementary effects is computed along a trajectory contained in a scaled and translated version of $Q_d$. 
Our work is concerned with

Morris clustered designs

Design matrices that allow computation of $m > 1$ elementary effects along each direction (i.e., each evaluation of $f$ is used to compute a larger number of $d_i$'s).

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

10 points in $Q_4$

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

(4, 2)-equitable subgraphs

7 points in $Q_4$
Why coming back to the problem?

Shortcomings of Morris clustered construction

- not guided by $m$
- cannot yield all possible values of $m$
- factored version (the most efficient) defined only when $d$ is not prime
- definition in the paper is not always equitable
- minimality of the size of the designs (efficiency) is not guaranteed.

Our contribution

Constructive algorithm for generation of the clustered designs of Morris method guided by the target value of $m$ and the dimension $d$ of the input space

- Handles generic values of $(d, m)$.
- Proovably equitable designs.
- For pairs $(d, m)$ for which Morris construction is defined, leads to designs of the same complexity.
Why studying problem 2?

Extend Morris Elementary Effects method to (cross) derivatives of second order

Elementary mixed-effects along directions $i, j \in \{1, \ldots, d\}$

$$d_{ij}^{(2)}(y) = \frac{1}{\Delta} [d_i(y + \Delta e_j) - d_i(y)], \quad i \in \{1, \ldots, d\}$$

Previous work

The new Morris Method, Campolongo & Braddock (Reliability Engineering and System Safety, 1999) : only defined for $c = 1$, less efficient designs than ours and no complete algorithmic construction.
How do we do it?

Two basic ideas

1. $(d, m)$-edge and $(d, c)$-cycle equitable subgraphs are recursively generated, by combining smaller equitable solutions (for smaller values of $d$, and $m$ or $c$)

2. use a polynomial representation to manipulate subgraphs and prove their properties
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Polynomial representation of subgraphs of $Q_d$

**Coding points of $Q_d$ by monomials**

$$s = \{s_1, s_2, \ldots, s_d\} \rightarrow \mathcal{P}_s(X_1, X_2, \ldots, X_d) = X_1^{s_1}X_2^{s_2} \ldots X_d^{s_d}$$

Example

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in Q_5 \rightarrow X_2X_3X_5 \in K(X_1, \ldots, X_5) = K_5$$

**Coding subgraphs of $Q_d$ by polynomials**

$$G \subset Q_d \rightarrow \mathcal{P}_G = \sum_{s \in G} \mathcal{P}_s$$

$\mathcal{P}_G$: degree at most one in each variable, coefficients in $\{0, 1\}$. 
Polynomial representation of subgraphs of $Q_d$

Example

\[ P = 1 + x_1 + x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 \subset Q_3 \]

Edge coloring of $Q_3$:

- Red: $x_1$
- Cyan: $x_2$
- Green: $x_3$
Polynomial representation of subgraphs of $Q_d$

Scalar product and structure

**Definition of $\langle \cdot, \cdot \rangle$**

$\mathcal{P}_s, \mathcal{P}_{s'}$ two monomials ($s, s' \in Q_d$)

Define the scalar product

$$\langle \mathcal{P}_s, \mathcal{P}_{s'} \rangle = 1_{s=s'}.$$

Extension to polynomials ($G, G' \subset Q_d$)

$$\langle \mathcal{P}_G, \mathcal{P}_{G'} \rangle = \sum_{s \in G, s' \in G'} \langle \mathcal{P}_s, \mathcal{P}_{s'} \rangle.$$

**Example**

$$\langle X_1X_2, X_1X_2 \rangle = 1, \quad \langle X_1X_2, X_1X_2X_3 \rangle = 0$$

$$\langle 1 + X_1 + X_2 + X_1X_2, 1 + X_1X_2 + X_3 \rangle = 2$$
Properties

- $\langle P_G, P_{G'} \rangle = |G \cap G'|$
- $\langle P_G, P_G \rangle = |G|$

Algebra over the polynomials

- **Addition** $+$ $\iff$ graph sum (nodes multiplicity may be $> 1$)
- **Multiplication** is defined modulo $X_i^2 = 1, i \in \{1, \ldots, d\}$

  Multiplication of $P_G$ by monomial $s = X_i$ $\iff$ reflection of $G$ along direction $i$

Example ($X_1$ corresponds to red edges)

\[
X_1(1 + X_1 + X_2 + X_1X_3 + X_2X_3) = X_1 + X_1^2 + X_1X_2 + X_1^2X_3 + X_1X_2X_3
\]

$\Rightarrow$

\[
X_1 + 1 + X_1X_2 + X_3 + X_1X_2X_3
\]
Problem reformulation in terms of polynomials

Facts:
1. Edges of color $i$ are preserved by multiplication by $X_i$. All other edges are moved elsewhere in $Q_d$
2. (Remember that $|G \cap G'| = \langle P_G, P_{G'} \rangle$)
3. $\Rightarrow$ the number of edges of $G$ of color $i$ is exactly $2 \langle P_G, X_i P_G \rangle$
4. $\Rightarrow$ the number of cycles in $G$ in colors $i, j$ is exactly $4 |P_G \cap X_i P_G \cap X_j P_G \cap X_i X_j P_G|$

Problem 1 reformulation

Optimal $(d, m)$-edge equitable designs are the solutions of

$$P^* = \arg\min_{P \in K_d} \langle P, P \rangle$$

s.t. $\langle P^*, X_i P^* \rangle = 2m$, $i \in \{1, 2, \ldots, d\}$.

We drop minimality, and assess the simpler problem of finding small $(d, m)$-edge equitable designs (not necessarily minimal).
Problem reformulation in terms of polynomials

Facts:

1. edges of color $i$ are preserved by multiplication by $X_i$. All other edges are moved elsewhere in $Q_d$
2. (remember that $|G \cap G'| = \langle P_G, P_{G'} \rangle$)
3. $\Rightarrow$ the number of edges of $G$ of color $i$ is exactly $2 \langle P_G, X_i P_G \rangle$
4. $\Rightarrow$ the number of cycles in $G$ in colors $i, j$ is exactly
   $4 |P_G \cap X_i P_G \cap X_j P_G \cap X_i X_j P_G|$

Problem 2 reformulation

Optimal $(d, c)$-cycle edge equitable designs are the solutions of

$$P^* = \arg \min_{P \in K_d} \langle P, P \rangle$$

s.t. $|P_G \cap X_i P_G \cap X_j P_G \cap X_i X_j P_G| = 4c, \quad i \neq j \in \{1, 2, \ldots, d\}.$

As for Problem 1, we relax the minimality constraint.
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   - Factored $(d, m)$-edge equitable designs
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Generation of \((d, m)\)-edge equitable subgraphs of \(Q_d\)

Recursive (in \(m\)) algorithm

**Initialisation**

- \(m = 1\), generic \(d\)

\[
G_d^1 = 1 + \sum_{i=1}^{d} X_1 \cdots X_i .
\]
Generation of \((d, m)\)-edge equitable subgraphs of \(Q_d\)

**Induction**

- \(m\) even

\[
G_d^m = G_{d-1}^{\frac{m}{2}} + X_1 X_d G_{d-1}^{\frac{m}{2}}
\]

**Example:**

\[
G_4^4 = G_3^2 + X_1 X_4 G_3^2
\]
Generation of \( (d, m) \)-edge equitable subgraphs of \( Q_d \)

**Induction**

- \( m \) odd

\[
G_d^m = G_{d-1}^{\frac{m-1}{2}} + X_1 X_d G_{d-1}^{\frac{m+1}{2}}
\]

**Example:**

\[
G_4^5 = G_3^2 + X_1 X_4 G_3^3
\]
**Theorem**

$G_d^m$ are $(d, m)$-edge equitable

*Proof:* use properties of scalar product (requires a condition on the solutions for consecutive values of $m$ that is guaranteed by the initialisation of the recursion)
Generation of \((d, m)\)-edge equitable subgraphs of \(Q_d\)

Topology and Initialisation

Other families of solutions can be obtained, by changing the initialization for small values of \(m\). This has an impact on the topology (and on the complexity) of the resulting designs.

\[ G_5^5, \text{Init } m = 1 \text{ only} \]

\[ G_5^5, \text{Init } m = 2, 3 \]
Factored \((d, m)\)-equitable designs

Direct application of our algorithm leads to less efficient designs than Morris when they are defined.

Factored application of our generic solution

\[
q_{\text{min}}(m) \triangleq \lceil \log_2(m) \rceil + 1 ,
\]

\[
d = (c - 1)q_{\text{min}}(m) + r , \quad r \in \{q_{\text{min}}(m), \ldots, 2q_{\text{min}}(m) - 1\} .
\]

\[
G_{\text{Morris}}(d, m) = G(q_{\text{min}}, m) + \sum_{j=1}^{c-2} (\text{Shift}_{jq_{\text{min}}} G(q_{\text{min}}, m) - 1) + \text{Shift}_{(c-1)q_{\text{min}}} G(r, m)
\]

Fully-defined and provably edge-equitable version of the basic idea of Morris factored designs.
Factored \((d, m)\)-edge equitable designs

Example

\[ G_4^{17} : \] 4 complete \(Q_3\) \((X_1 \cdots X_3, X_4 \cdots X_6, X_7 \cdots X_9, X_{10} \cdots X_{12})\),

\[ \text{together with } G_5^4 \text{ (over } X_{13} \cdots X_{17} \text{)} \]
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Some notation

\[ \text{Line}(X_1, \ldots, X_d) = \sum_{i=1}^{d} \prod_{j \leq i} X_j \]

\[ \text{Circle}(X_1, \ldots, X_d) = \text{Line}(X_1, \ldots, X_d) + \left( \prod_{j=1}^{d} X_j \right) \text{Line}(X_1, \ldots, X_d) \]

\[ \text{Bubble}((X_1, \ldots, X_d)) = \text{Polynomial in the } d \text{ variables with 3 edges of each colour} \]
(d, 1)-cycle equitable subgraphs

Initialisation
For \( d = 2 \) and \( c = 1 \), define \( H^1_2 = Q_2 \)

Induction
For \( d > 2 \) and \( c = 1 \), define \( H^1_d = H^1_{d-1} + X_d (1 + \text{Line}(X_1, \ldots, X_{d-1})) \)
(\(d, 2\))-cycle and (\(d, 3\))-cycle equitable subgraphs (\(H^d_2, H^d_3\))

**Initialisation**

For \(d = 3\) and \(c = 2\), define \(H^3_2 = Q_3\)

For \(d = 4\) and \(c = 3\), define \(H^4_3 = Q_4 - X_2X_4\)

**Induction**

For \(d > 3\) and \(c = 2\), define \(H^d_2 = H^{d-1}_2 + X_d\text{Circle}(X_1, \ldots, X_{d-1})\)

For \(d > 4\) and \(c = 3\), define \(H^d_3 = H^{d-1}_3 + X_d\text{Bubble}(X_1, \ldots, X_{d-1})\)

![Circle(4)](image1)

![Bubble(6)](image2)
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Size of the design

- If initialization for $m = 1$, 
  \[ |G_m^d| = m(d - \kappa) + 2^{\kappa+1} - m \]
  where $\kappa = \lfloor \log_2(m) \rfloor$.
- We derived a closed formula $|G_m^d|$ for initialization at $m = 2, 3$
  \[ |G_m^d| = c(m) + \alpha(m)d \]
  Size of factored solution is also known exactly.
- We also have a closed formula for $|H_c^d|$. 
Economy

Definition

Morris index, ($|G_d^m|$ should be small $\Leftrightarrow \chi$ large)

\[
\text{Economy: } \chi = \frac{\text{total \# elementary effects}}{|G_d^m|} = \frac{md}{|G_m^d|}
\]

Economy of the $(d, m)$-edge equitable designs

Evolution of $\chi$ as $d$ grows, $m = 10$.
Factored designs, designs with init $G_1^d$, and with init $G_2^d, G_3^d$. 
Size of the \((d, c)\)-cycle equitable designs

We obtain:

<table>
<thead>
<tr>
<th>(c)</th>
<th>Nb Edges</th>
<th>Nb Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(d)</td>
<td>(\frac{d^2 + d + 2}{2})</td>
</tr>
<tr>
<td>2</td>
<td>(2d - 4)</td>
<td>(d^2 - d + 2)</td>
</tr>
<tr>
<td>3</td>
<td>(3d - 5)</td>
<td>(\frac{3d^2 - 7d + 10}{2})</td>
</tr>
</tbody>
</table>

For random designs and *New Morris* designs

<table>
<thead>
<tr>
<th>(c)</th>
<th>Nb Edges</th>
<th>Nb Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2\binom{d}{2})</td>
<td>(4\binom{d}{2})</td>
</tr>
<tr>
<td>2</td>
<td>(4\binom{d}{2})</td>
<td>(8\binom{d}{2})</td>
</tr>
<tr>
<td>3</td>
<td>(6\binom{d}{2})</td>
<td>(12\binom{d}{2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c)</th>
<th>Nb Edges</th>
<th>Nb Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>not edge equitable</td>
<td>(4 \ d^2 - d + 2)</td>
</tr>
<tr>
<td>2</td>
<td>(\star)</td>
<td>(\star)</td>
</tr>
<tr>
<td>3</td>
<td>(\star)</td>
<td>(\star)</td>
</tr>
</tbody>
</table>
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Morris example function

\[ f(x) = \beta_0 + \sum_{i=1}^{20} \beta_i w_i + \sum_{i<j}^{20} \beta_{ij} w_i w_j + \sum_{i<j<l}^{5} \beta_{ijl} w_i w_j w_l + \sum_{i<j<l<s}^{4} \beta_{ijls} w_i w_j w_l w_s \]

\[ w_i = 2X_i - 1, \quad i \in \{1, 2, 4, 6, 8, \ldots, 20\}, \quad w_i = 2.2X_i/(X_i + 0.1) - 1, \quad i \in \{3, 5, 7\}. \]

\[ \beta_i = 20, \quad i \in \{1, \ldots, 10\}, \quad \beta_{ij} = -15, \quad i, j \in \{1, \ldots, 6\} \]

\[ \beta_{ijl} = -10, \quad i, j, l \in \{1, \ldots, 5\}, \quad \beta_{ijls} = 5, \quad i, j, l, s \in \{1, \ldots, 4\}. \]

Remaining 1\textsuperscript{st} and 2\textsuperscript{nd} order coefficients are independent realisations of a standard normal distribution, \( \beta_i \sim \mathcal{N}(0, 1), \quad i \notin \{1, \ldots, 10\}, \beta_{ij} \sim \mathcal{N}(0, 1), \quad i, j \notin \{1, \ldots, 6\}. \) For this function the relevant classes of input factors are

\[ C_{\text{irrelevant}} = \{11, \ldots, 20\}, \quad C_{\text{linear}} = \{8, 9, 10\}, \quad C_{\text{other}} = \{1, \ldots, 7\}. \]

Note: \( X_7 \) is a purely non-linear term, while \( X_6 \) is an interaction factor.
Screening of Morris example function \((m = 4, r = 3)\)

Total number of derivatives per direction: 12

About half the number of function evaluations compared to \(m = 1\).
Study of cross derivatives

Analysis concentrated on smaller class $\mathcal{C}_{\text{other}}$
We detect that $X_7$ as a **non-linear factor with no interaction** with the other factors as well as the **bilinear term** $X_2X_6$. 
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Up to now

1. Recursive algorithm for \((d, m)\)-edge equitable graphs that completes the definition of clustered Morris designs.

2. Recursive algorithm for \((d, c)\)-cycle equitable graphs for \(c = 1, 2, 3\) (can be exploited to build the skeleton of the FANOVA graph).

3. Explicit formulas for the size of the designs.

4. Uses polynomial representation of subgraphs of \(Q_d\) and an appropriate definition of inner product as formal tools.

5. Polynomial representation enables direct identification of pairs of design points involved in the derivatives along each direction (or pairs of directions, for mixed effects).
Further work

Open issues ...

- minimality (of factored designs)?
- effect of initialization
- relation to other classes of subgraphs of the hypercube (median graphs, mesh graphs, ...)
- Generalize to subgraphs of $\{0, 1, \ldots, k\}^d$ for detection of higher order effects in each input factor
Generation of \((d, m)\)-equitable subgraphs of \(Q_d\)
Demonstration (equitable designs)

\(m\) even. Assume \(G_{d-1}^{m/2}\) is \((d - 1, m)\)-equitable.

\[
\langle G_d^m, X_i G_d^d \rangle = \begin{cases} 
\langle G_{d-1}^{m/2}, X_i G_{d-1}^{m/2} \rangle + \\
\langle X_1 X_d G_{d-1}^{m/2}, X_i X_1 X_d G_{d-1}^{m/2} \rangle = 2m, & \text{if } i < d \\
\langle G_{d-1}^{m/2}, X_1 G_{d-1}^{m/2} \rangle + \\
\langle X_1 X_d G_{d-1}^{m/2}, X_1 G_{d-1}^{m/2} \rangle = 2m, & \text{if } i = d
\end{cases}
\]
Generation of \((d, m)\)-equitable subgraphs of \(Q_d\)

Proof (equitable designs)

**m odd.** Assume \(G_{\frac{m-1}{2}}\) and \(G_{\frac{m+1}{2}}\) equitable

\[
\langle G_d^m, X_i G_d^m \rangle = \begin{cases} 
\langle G_{\frac{m-1}{2}}, X_i G_{\frac{m-1}{2}} \rangle + \\
\quad + \langle G_{\frac{m+1}{2}}, X_i G_{\frac{m+1}{2}} \rangle, & \text{if } i < d \\
2 \langle G_{\frac{m-1}{2}}, X_1 G_{\frac{m+1}{2}} \rangle, & \text{if } i = d
\end{cases}
\]

\[
= \begin{cases} 
(m - 1) + (m + 1) = 2m, & \text{if } i < d \\
2 \langle G_{\frac{m-1}{2}}, X_1 G_{\frac{m+1}{2}} \rangle, & \text{if } i = d\end{cases}
\]

Thus

\[G_d^m\] is \((d, m)\)-equitable \iff \(\langle G_{\frac{m-1}{2}}, X_1 G_{\frac{m+1}{2}} \rangle = m\)

It can be shown that

\[
\langle G_{\frac{k-1}{2}}, X_1 G_{\frac{k-1}{2}} \rangle = 2k - 1 \Rightarrow \langle G_{\frac{2k-1}{2}}, X_1 G_{\frac{2k}{2}} \rangle = 4k - 1
\]

\[
\langle G_{\frac{k+1}{2}}, X_1 G_{\frac{k+1}{2}} \rangle = 2k + 1 \Rightarrow \langle G_{\frac{2k}{2}}, X_1 G_{\frac{2k+1}{2}} \rangle = 4k + 1
\]
Generation of \((d, m)\)-equitable subgraphs of \(Q_d\)

Demonstration

\[
\langle G_{d-1}^k, X_1 G_{d-1}^{k+1} \rangle = 2k + 1
\]

Check that is true for \(k = 1\), using the construction \(G_d^2\).

\[
\langle G_d^1, X_1 G_d^2 \rangle = \left( \langle 1 + \sum_{i=1}^{d} X_1 \cdots X_i \rangle, (X_1 + X_d)(1 + \sum_{j=1}^{d-1} X_1 \cdots X_j) \right) \\
= \langle 1, 1 \rangle + \langle X_1, X_1 \rangle + \langle X_1 \cdots X_d, X_1 \cdots X_d \rangle \\
= 3
\]

The identity is thus valid for all \(k\), completing the proof that our algorithm generates \((d, m)\)-equitable subgraphs of \(Q_d\).
Morris designs

\[ \mathbb{R}^d = \prod_{j=1}^{t} \mathbb{R}^q, \quad d = tq \quad Y = \bigcup_{j=1}^{t} Y^j, \]

where

\[ Y^j = v_j + C \left[ \underbrace{O_q \cdots O_q}_{j-1 \text{ blocks}} I_q \underbrace{O_q \cdots O_q}_{t-j \text{ blocks}} \right], \quad j = 1, \ldots, t, \]

\[ B_M = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
C & O & O & \cdots & 0 \\
J & C & O & \cdots & 0 \\
J & J & C & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
J & J & J & \cdots & C
\end{bmatrix} \]

0: \( q \)-element (row) vector of zeros, \( J \): \( n_C \times q \) matrix of ones.
Morris designs

\[ d = 9 = 3 \times 3 \]

\[
\begin{bmatrix}
C & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
J & C & 0 \\
\end{bmatrix}
\begin{bmatrix}
J & J & C \\
\end{bmatrix}
\]

\[ \{X_1 \cdots X_3\} \quad \{X_4 \cdots X_6\} \quad \{X_7 \cdots X_9\} \]
Morris designs

Choice of $C$

Chose $\mathcal{I} \subset \{1, \ldots, q\}$. Let the rows of $C$ (of dimension $n_C \times q$) be the set of all binary vectors with $\ell$ entries equal to one, $\forall \ell \in \mathcal{I}$.

$$n_C = \sum_{\ell \in \mathcal{I}} C^q_{\ell}$$

$$m(\mathcal{I}) = l(1)l(q) + \sum_{j=2}^{q} l(j-1)l(j)C_{j-1}^{q-1}$$

Size of Morris designs

$$n_M = tn_C + 1 = \frac{d}{q} \sum_{\ell \in \mathcal{I}} C^q_{\ell} + 1$$
Initialisation

\[ m = 2 \ d \text{ odd} \]

\[ m = 2, \ d \text{ even} \]
Initialisation

\[ m = 3 \]