Roe Solver and Entropy Corrector for Hyperbolic Systems with Uncertain Coefficients

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Introduction

Parametric Uncertainty Quantification

- uncertainties in input quantities (model parameters, initial and boundary conditions)
- uncertain quantities parametrized by random variables with known distribution functions

Stochastic spectral methods

- decompose random quantities on suitable approximation bases (Ghanem and Spanos 91)
- Stochastic expansion of the solution:

\[ U(x, t, \xi) \approx U^P(x, t, \xi) = \sum_{\alpha=1}^{P} u_\alpha(x, t) \psi_\alpha(\xi). \]

\( u_\alpha(x, t) \) stochastic modes of the solution.
Different computational strategies

- **probabilistic collocation**: stochastic modes evaluated by polynomial interpolation
- **non-intrusive projection**: stochastic modes evaluated by numerical integration
- **stochastic Galerkin**: reformulated deterministic problem for the stochastic modes

**Stochastic Galerkin** methods:
- rely on a **weak form** of the problem
- well suited for **mathematical analysis**
- design of **adaptive methods**
Euler equations (Sod Shock Tube)

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0,
\]

\[
U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E + p))^T,
\]

\[
v = \frac{q}{\rho}, \quad p = (\gamma - 1) \left( E - \frac{1}{2} \rho v^2 \right).
\]

\[
\gamma(\xi) = 1.4 + 0.2 \xi, \quad \xi \sim U[0, 1].
\]
Euler equations (Sod Shock Tube)

Stochastic density $\rho(x, t, \xi)$ at $t = 0.25$ and $t = 3.25$.

Two main difficulties:

- solutions discontinuous in physical as well as in stochastic spaces
- nonlinearities in the flux functions
State of the art

Stochastic spectral methods applied to a large variety of engineering problems (elasticity, thermal science, fluid flows, chemical/biological systems,...) governed by elliptic, parabolic, ODE or incompressible NS models.

Hyperbolic models

- non-intrusive approaches: multi-element probabilistic collocation methods (Lin et al. 08)
- pseudo-intrusive methods: stochastic modes of flux computed by quadrature methods (Ge et al. 08, Poette et al. 09)
- fully intrusive methods: scalar linear wave equation (Gottlieb and Xiu 08), mean flux upwinding (Lin et al. 06)
Objectives

- Develop fully intrusive stochastic Galerkin method
- Investigate hyperbolicity of the Galerkin system
- Design a Roe-type solver with entropy corrector
- Account for non-smooth solutions
1. **Galerkin projection**
   - Stochastic hyperbolic systems
   - Stochastic discretization
   - The Galerkin system
   - Hyperbolicity of the Galerkin system

2. **Numerical method**
   - Numerical scheme
   - Roe linearized matrix
   - Absolute value of a matrix
   - The upwind scheme

3. **Results**
   - Periodic Burgers equation
   - Euler equations
   - Entropy corrector
1 Galerkin projection
Stochastic hyperbolic systems
Stochastic discretization
The Galerkin system
Hyperbolicity of the Galerkin system

2 Numerical method

3 Results
Stochastic parametrization

\[ \xi = (\xi_1, \ldots, \xi_N) \sim \mathcal{U}(\Xi = [0, 1]^N) \rightarrow L^2(\Xi) \]

The corresponding space of the second-order random variables with the expectation operator \[ \langle H \rangle = \int_{\Xi} H(y)dy. \]

Stochastic hyperbolic systems

We seek for \( U(x, t, \xi) \in \mathbb{R}^m \otimes L^2(\Xi, \mu_{\xi}) \) solving a.s.

\[
\begin{cases}
\frac{\partial}{\partial t} U(x, t, \xi) + \frac{\partial}{\partial x} F(U(x, t, \xi); \xi) = 0, \\
U(t = 0, x, \xi) = U_0(x, \xi).
\end{cases}
\]

\((x, t, \xi) \in \Omega \times [0, T] \times \Xi,\)

\(\nabla_U F \in \mathbb{R}^{m,m} \) stochastic Jacobian matrix \(\mathbb{R}\)-diagonalizable almost surely.
Stochastic discretization

We approximate $U(x, t, \xi)$ in the stochastic space of fully tensorized piecewise polynomial functions $S^{N_0, N_r}$:

- $N_r$: resolution level,
- $N_0$: expansion order.

**Remark:** Also possible to work with smaller stochastic approximation spaces, using for instance sparse tensorization.

**Case $N = 1$.

Exemple pour $N_r = N_0 = 3$ : $U(\xi) \in S^{3,3}$. $U(\xi)$

![Graph](graph.png)
Stochastic discretization

We approximate $U(x, t, \xi)$ in the stochastic space of fully tensorized piecewise polynomial functions $S^{N_{0}, N_{r}}$:

- $N_{r}$: resolution level,
- $N_{0}$: expansion order.

Case $N = 1$.

$$\dim S^{N_{0}, N_{r}} = (N_{0} + 1)2^{N_{r}} =: P_{\pi}P_{\sigma} := P.$$ 

Select the Stochastic Element (SE) orthonormal basis

$$\{\Psi_{\alpha}\}_{\alpha=1,...,P},$$

which corresponds to local (rescaled) Legendre polynomial bases, s.t.

$$\text{span}(\Psi_{1}, \ldots, \Psi_{P}) = S^{N_{0}, N_{r}}.$$

$\alpha = (\alpha_{\sigma}, \alpha_{\pi})$, where $\alpha_{\sigma}$ refers to the stochastic element and $\alpha_{\pi}$ to the polynomial function in the stochastic element.
**Stochastic discretization**

**General case** $N > 1$ obtained by **full tensorization**.

\[
\text{dim } S^{\text{No}, \text{Nr}} = (\text{No} + 1)^N 2^{\text{NNr}} =: P_{\pi} P_{\sigma} := P.
\]

Select the **Stochastic Element (SE)** orthonormal basis \( \{\psi_\alpha\}_{\alpha=1,\ldots,P} \), which corresponds to local fully tensorized (rescaled) Legendre polynomial bases, s.t.

\[
\text{span}(\psi_1, \ldots, \psi_P) = S^{\text{No}, \text{Nr}}.
\]

\( \alpha = (\alpha_\sigma, \alpha_\pi) \), where \( \alpha_\sigma \) refers to the stochastic element and \( \alpha_\pi \) to the polynomial function in the stochastic element.

**Stochastic expansion of the solution**:

\[
U(x, t, \xi) \approx U^P(x, t, \xi) = \sum_{\alpha=1}^{P} u_\alpha(x, t) \psi_\alpha(\xi).
\]

\( u_\alpha(x, t) \in \mathbb{R}^m \) stochastic modes of the solution.
The Galerkin system

Galerkin projection of the original stochastic problem:

\[
\begin{cases}
\langle \psi_\alpha \frac{\partial U^P}{\partial t} \rangle + \langle \psi_\alpha \frac{\partial F(U^P; \cdot)}{\partial x} \rangle = 0, \quad \forall \alpha = 1, \ldots, P, \\
\langle \psi_\alpha U^P \rangle (t = 0) = \langle \psi_\alpha U_0(x, \cdot) \rangle, \quad \forall \alpha = 1, \ldots, P.
\end{cases}
\]

We seek for \( u(x, t) \) solving

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u(x, t)) = 0, \quad u(x, t = 0) = u^0(x),
\]

\[
u(x, t) = (u_1(x, t), \ldots, u_P(x, t))^T \in \mathbb{R}^{mP},
\]

\[
u^0(x) = (\langle \psi_\alpha U^0 \rangle),
\]

\[
f(u(x, t)) = (f_1(u), \ldots, f_P(u))^T \in \mathbb{R}^{mP},
\]

\[
f_\alpha(u) := \langle \psi_\alpha F(U^P; \cdot) \rangle, \quad \alpha = 1, \ldots, P.
\]
Hyperbolicity of the Galerkin system

\[
(\nabla u f(u))_{\alpha,\beta=1,\ldots,P} = \langle \nabla u F(U^P; \cdot) \psi_{\alpha} \psi_{\beta} \rangle_{\alpha,\beta=1,\ldots,P}.
\]

Is the Galerkin system hyperbolic?

\[\iff \nabla u f(u) \in \mathbb{R}^{mP,mP} \text{ -diagonalizable?} \]

Diagonal block structure owing to the decoupling of the problem over different stochastic elements:

\[
\nabla u f(u) = \begin{pmatrix}
[\nabla u f]^1 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & [\nabla u f]^\alpha_{\sigma} & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & [\nabla u f]^{P_{\sigma}}
\end{pmatrix}.
\]

\[\rightarrow \text{ It suffices to consider } P_{\sigma} = 1 \text{ and } \nabla u f \text{ of size } mP_{\pi}.\]
Hyperbolicity of the Galerkin system

HYPERBOLICITY proven in two specific cases:

**Theorem**

HYPERBOLICITY, if the original stochastic system is a scalar conservation law.

**Theorem**

HYPERBOLICITY, if the stochastic Jacobian matrix $\nabla_u F(\cdot; \xi)$

- is either symmetric (almost surely)
- or its eigenvectors are deterministic (independent of the uncertainty).

Applications: Scalar wave equation with uncertain sound velocity. / Linear hyperbolic systems with uncertainty only on initial or boundary conditions.
Hyperbolicity of the Galerkin system

In the general case, we consider the approximate Galerkin Jacobian matrix $\overline{\nabla u f}$ whose coefficients are obtained by approaching the coefficients of $\nabla u f$ by a Gauss quadrature

$$
\left( \overline{\nabla u f(u)} \right)_{\alpha\pi,\beta\pi=1,\ldots,P\pi} = 
\left( \sum_{\gamma=1}^{P\pi} \omega_{\gamma} \nabla u F(U^P(\xi_{\gamma}); \xi_{\gamma}) \psi_{\alpha\pi}(\xi_{\gamma}) \psi_{\beta\pi}(\xi_{\gamma}) \right)_{\alpha\pi,\beta\pi=1,\ldots,P\pi},
$$

$\{\xi_{\gamma}\}_{\gamma=1,\ldots,P\pi}$ set of the Gauss points, with associated weights $\{\omega_{\gamma}\}_{\gamma=1,\ldots,P\pi}$.
Hyperbolicity of the Galerkin system

Assume $\nabla_u F(U^p(\xi); \xi) = L(\xi) \Lambda(\xi) R(\xi)$,

**Theorem**

$\nabla_u f(u)$ is $\mathbb{R}$-diagonalizable with eigenvalues $\{\lambda'_\gamma\}_{\gamma=1,\ldots,m\pi}$ and right and left eigenvectors $\{r'_\gamma\}_{\gamma=1,\ldots,m\pi}$ and $\{l'_\gamma\}_{\gamma=1,\ldots,m\pi}$ given by

$$
\begin{cases}
\{\lambda'_\gamma\}_{\gamma=1,\ldots,m\pi} = \lambda'_{k\eta} = \Lambda^k(\xi_\eta), \\
\{r'_\gamma\}_{\gamma=1,\ldots,m\pi} = (r'_{k\eta})_\beta = \left(\varpi_\eta R^k(\xi_\eta) \psi_\beta(\xi_\eta)\right)_\beta, \\
\{l'_\gamma\}_{\gamma=1,\ldots,m\pi} = (l'_{k\eta})_\beta = \left(\varpi_\eta L^k(\xi_\eta) \psi_\beta(\xi_\eta)\right)_\beta,
\end{cases}
$$

where $k = 1, \ldots, m$, $\eta = 1, \ldots, P_\pi$, $\beta = 1, \ldots, P_\pi$.

$\rightarrow \{\lambda'_\gamma\}_{\gamma=1,\ldots,m\pi}$, $\{r'_\gamma\}_{\gamma=1,\ldots,m\pi}$, and $\{l'_\gamma\}_{\gamma=1,\ldots,m\pi}$ approximations of the eigenvalues and eigenvectors of $\nabla_u f$. 

1. Galerkin projection

2. Numerical method
   - Numerical scheme
   - Roe linearized matrix
   - Absolute value of a matrix
   - The upwind scheme

3. Results

Outline

1. Galerkin projection
2. Numerical method
3. Results
Numerical scheme

Discretization of the Galerkin system using a FV method:

\[
\begin{align*}
    u_{i}^{n+1} &= u_{i}^{n} - \frac{\Delta^n t}{\Delta x} \left( \varphi(u_{i}^{n}, u_{i+1}^{n}) - \varphi(u_{i-1}^{n}, u_{i}^{n}) \right),
\end{align*}
\]

- \(\Delta x\) (uniform) spatial step,
- \(\Delta^n t\) time step,
- \(\varphi(\cdot, \cdot)\) 1\(^{st}\) order numerical flux function:

\[
\varphi(u_{i}^{n}, u_{i+1}^{n}) = \frac{f(u_{i}^{n}) + f(u_{i+1}^{n})}{2} - \left| a(u_{i}^{n}, u_{i+1}^{n}) \right| \frac{u_{i+1}^{n} - u_{i}^{n}}{2}.
\]

centered part of the flux \hspace{1cm} \text{upwind matrix}

chosen as explained below
Roe linearized matrix

- Assume that the original stochastic problem possesses a Roe linearized matrix and a Roe state a.s.,
  \[(U_L, U_R) \rightarrow A^{\text{Roe}}(U_L, U_R) = \nabla_U F(U^{\text{Roe}}_{LR}; \cdot).\]

- Given two states \(u_L\) and \(u_R\) of the Galerkin system,
  \[
  \begin{align*}
  u_L &\rightarrow U^P_L \\
  u_R &\rightarrow U^P_R \\
  \end{align*}
  \rightarrow U^{\text{Roe}}_{LR},
  \]
  \[
  \rightarrow a^{\text{Roe}}(u_L, u_R) = \langle \nabla_U F(U^{\text{Roe}}_{LR}; \cdot) \psi_\alpha \psi_\beta \rangle.
  \]

Theorem
\(a^{\text{Roe}}\) is a Roe linearized matrix for the Galerkin system.

Choice of upwinding:
\[
\varphi(u^n_i, u^n_{i+1}) = \frac{f(u^n_i) + f(u^n_{i+1})}{2} - |a^{\text{Roe}}(u^n_i, u^n_{i+1})| \frac{u^n_{i+1} - u^n_i}{2}.
\]

- Consistency of the numerical scheme
- Conservativity through shocks
Efficient approximation of the absolute value of a matrix

A deterministic $\mathbb{R}$-diagonalizable matrix of size $N_A$. Known data: $\{\lambda_i\}_{i=1,...,N_A}$ eigenvalues of $A$.

\[
A = \sum_{i=1}^{N_A} \lambda_i l_i \otimes r_i, \quad |A| = \sum_{i=1}^{N_A} |\lambda_i| l_i \otimes r_i.
\]

For a polynomial $q$

\[
q(A) = \sum_{i=1}^{N_A} q(\lambda_i) l_i \otimes r_i.
\]

→ Determination of a polynomial $q_d,\{\lambda_i\}$ with low degree $d \ll N_A$ which minimizes the least-square error

\[
\sum_{i=1}^{N_A} (|\lambda_i| - q_d,\{\lambda_i\}(\lambda_i))^2.
\]

In fact, determination of $q_d,\{\lambda'_i\}(A)$ from $\{\lambda'_i\}_{i=1,...,N_A}$ approximate eigenvalues of $A$.

→ $|A| \approx q_d,\{\lambda'_i\}(A)$. 
**The upwind scheme**

At each interface $LR$ in physical space,

\[
 u_{i,i+1}^{\text{Roe}} := \left( \left\langle \psi \alpha \, U_{i,i+1}^{\text{Roe}} \right\rangle \right)_{\alpha=1,\ldots,P},
\]

projected Roe state in $S^{\text{No}, \text{Nr}}$.

**Parallelisation** of the procedure on each stochastic element $\alpha_\sigma$, $1 \leq \alpha_\sigma \leq P_\sigma$,

→ Evaluate approximate eigenvalues \( \{ \lambda'_\gamma \}_{\gamma=1,\ldots,mP_\pi} \)

\[
\{ \lambda'_\gamma \}_{\gamma=1,\ldots,mP_\pi} = \text{spec}(\nabla u f(u_{i,i+1}^{\text{Roe}})).
\]

→ Determine the local polynomial \( q_d, \{ \lambda'_\gamma \} \) fitting \( \{ \lambda'_\gamma \}_{\gamma=1,\ldots,mP_\pi} \).

→ Approximate the absolute value of \( a^{\text{Roe}}(u^n_i, u^n_{i+1}) \)

\[
|a^{\text{Roe}}(u^n_i, u^n_{i+1})| \approx q_d,\{ \lambda'_\gamma \}(\nabla u f(u_{i,i+1}^{\text{Roe}})).
\]
The upwind scheme

\[ u_i^{n+1} = u_i^n - \frac{\Delta^n t}{\Delta x} (\varphi(u_i^n, u_{i+1}^n) - \varphi(u_{i-1}^n, u_i^n)) , \]

where the numerical flux \( \varphi(u_i^n, u_{i+1}^n) \) is computed in this way

\[ \varphi(u_i^n, u_{i+1}^n) = \frac{f(u_i^n) + f(u_{i+1}^n)}{2} - q_d,\{\lambda'\} (\nabla f(u_{i,i+1}^{\text{Roe}})) \frac{u_{i+1}^n - u_i^n}{2}. \]

CFL condition :

\[ \frac{\Delta^n t^{\alpha \sigma}}{\Delta x} = \frac{\text{CFL}}{\max_{LR \in I, \gamma = 1, \ldots, mP\pi} |\lambda'_{\gamma}|} , \quad \Delta^n t = \min_{1 \leq \alpha \sigma \leq P\sigma} \Delta^n t^{\alpha \sigma}. \]
1. Galerkin projection

2. Numerical method

3. Results
   - Periodic Burgers equation
   - Euler equations
   - Entropy corrector
Periodic Burgers equation

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},
\]

initial random shock locations:

\[X_{1,2} = 0.1 + 0.1 \xi_1, \quad X_{2,3} = 0.3 + 0.1 \xi_2, \quad \xi_1, \xi_2 \sim \mathcal{U}[0, 1].\]
Periodic Burgers equation

Stochastic solution $U(x, t, (\xi_1, \xi_2))$ at observation point $x_0(t) = 0.25 + 0.5t$ as a function of $(\xi_1, \xi_2)$ and for different times. Computations with $No = Nr = 3$. 
### Periodic Burgers equation

<table>
<thead>
<tr>
<th>Expectation</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Expectation Diagram" /></td>
<td><img src="image2.png" alt="Standard Deviation Diagram" /></td>
</tr>
</tbody>
</table>

Space-time diagrams of the expectation $\langle U(x, t, \cdot) \rangle$ and standard deviation $\sigma(U(x, t, \cdot))$ of the stochastic solution. $N_0 = N_r = 3$. 
### Burgers equation

<table>
<thead>
<tr>
<th>Nr = 3, No = 1</th>
<th>Nr = 3, No = 2</th>
<th>Nr = 3, No = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph 1" /></td>
<td><img src="image2.png" alt="Graph 2" /></td>
<td><img src="image3.png" alt="Graph 3" /></td>
</tr>
<tr>
<td>Nr = 1, No = 3</td>
<td>Nr = 2, No = 3</td>
<td>Nr = 4, No = 3</td>
</tr>
<tr>
<td><img src="image4.png" alt="Graph 4" /></td>
<td><img src="image5.png" alt="Graph 5" /></td>
<td><img src="image6.png" alt="Graph 6" /></td>
</tr>
</tbody>
</table>

Stochastic solution of the Burgers equation as a function of $(\xi_1, \xi_2)$ at $x = 0.5$ and $t = 0.5$ for different Nr and No.
\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \\
U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E + p))^T, \\
v = \frac{q}{\rho}, \quad p = (\gamma - 1) \left( E - \frac{1}{2} \rho v^2 \right).
\]

\[\gamma(\xi) = 1.4 + 0.2 \xi, \quad \xi \sim U[0, 1].\]

**Computation of the Galerkin flux**: using a pseudo-spectral approximation (Debusschere et al. 04)

- \( a \times b \approx a \ast b = \sum_{\alpha=0}^{P} (a \ast b)_{\alpha} \psi_{\alpha}, \)
  \([a \ast b]_{\alpha} = \sum_{\beta, \delta=0}^{P} a_{\beta} b_{\delta} M_{\alpha\beta\delta}, \quad M_{\alpha\beta\delta} = \langle \psi_{\alpha} \psi_{\beta} \psi_{\gamma} \rangle\)
- \(1/a \approx a^{-*}\) obtained by solving \(a \ast a^{-*} = 1\)
- \(p \approx (\gamma - 1) \ast (E - (q \ast q) \ast (1/\rho))/2\)
- \(\sqrt{a} \approx a^{*}/2\) obtained by solving \((a^{*}/2) \ast (a^{*}/2) = a\)
Euler equations

<table>
<thead>
<tr>
<th>Deterministic problem ($\gamma = 1.5$)</th>
<th>Expected density</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Space-time diagrams of the deterministic density for $\gamma = 1.5$, the expected density, and the standard deviations in the density for early and longer times. $N_r = 3$ and $N_o = 2$." /></td>
<td></td>
</tr>
</tbody>
</table>

Space-time diagrams of the deterministic density for $\gamma = 1.5$, the expected density, and the standard deviations in the density for early and longer times. $N_r = 3$ and $N_o = 2$. 
Euler equations

Stochastic density as a function of $(x, \xi)$. $Nr = 3$ and $No = 2$. 
Normalized computational times $T_{CPU}$ for different stochastic discretization parameters $Nr$ and $No$. $Nc = 250$.

→ computational costs scale as $dim S^{No,Nr}$ at least for moderate $No$. 

<table>
<thead>
<tr>
<th>$Nr$</th>
<th>$No$</th>
<th>$T_{CPU}$</th>
<th>$dim S^{Nr,No}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>4.0</td>
<td>(4)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6.9</td>
<td>(8)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11.8</td>
<td>(12)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>17.1</td>
<td>(16)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>24.8</td>
<td>(20)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>8.1</td>
<td>(8)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>13.9</td>
<td>(16)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>23.2</td>
<td>(24)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>34.1</td>
<td>(32)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>49.3</td>
<td>(40)</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>16.1</td>
<td>(16)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>27.8</td>
<td>(32)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>46.5</td>
<td>(48)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>68.1</td>
<td>(64)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>98.0</td>
<td>(80)</td>
</tr>
</tbody>
</table>
Entropy corrector

Euler equations (Sod Shock Tube)

\[ \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \]

\[ U = (\rho, q, E)^T, \quad F(U) = (\rho v, \rho v^2 + p, v(E + p))^T, \]

\[ v = \frac{q}{\rho}, \quad p = (\gamma - 1) \left( E - \frac{1}{2}\rho v^2 \right). \]

\[ Ma^0(\xi) = \begin{cases} 
0.7 + 0.5 \xi, & \xi \in [0, 1/4], \\
2.46 \times (0.7 + 0.5 \xi), & \xi \in ]1/4, 1], 
\end{cases} \]

\[ \xi \sim U[0, 1]. \]
Entropy corrector

Euler equations (Sod Shock Tube)

Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without entropy corrector (using $N_r = 3$ and $N_o = 2$):

Entropy-violating shock! $\rightarrow$ Need for an entropy corrector!
Entropy corrector

Non-parametrized entropy corrector proposed by Dubois and Mehlmann (96) for Roe solver in the deterministic case

- Avoid entropy-violating shocks
- Nonlinear modification of the numerical flux in the vicinity of sonic points
- Detection of sonic expansion waves based on reconstruction of intermediate states for each couple of left and right states and test on sign of eigenvalues of the Roe linearized matrix

→ Adaptation to the present context
Euler equations (Sod Shock Tube)

Stochastic density $\rho(x, t, \xi)$ at $t = 1$ obtained without (left) and with (right) the entropy corrector using $N_r = 3$ and $N_o = 2$.

Comparison of the mean and standard deviation of the numerical density at $t = 1$, computed with a Galerkin method (using $N_r = 3$ and $N_o = 2$) and a MC method.
Some details

- **Parallelisation** of the procedure on each stochastic element $\alpha_\sigma$, $1 \leq \alpha_\sigma \leq P_\sigma$

- Compute the $mP_\pi$ characteristic variables $\{\beta'_\gamma\}_{\gamma=1,...,mP_\pi}$
  \[
  u_L - u_R \approx \sum_{\gamma=1}^{mP_\pi} \beta'_\gamma r'_\gamma(u^\text{Roe}_{LR}).
  \]

- **Reconstruct** the $mP_\pi$ intermediate states at each physical interface
  \[
  u'_{\gamma} = u'_{\gamma-1} + \beta'_\gamma r'_\gamma(u^\text{Roe}_{LR}).
  \]

- Determine the set of sonic indices:
  \[
  S' = \{\gamma, \lambda'_\gamma(u'_{\gamma-1}) < 0 < \lambda'_\gamma(u'_{\gamma})\}.
  \]

- The indexing of $\{\lambda'_\gamma\}_\gamma$ and $\{r'_\gamma\}_\gamma$, $\gamma = 1, \ldots, mP_\pi$, provides a correspondence between approximate eigenvalues and eigenvectors and is central to determine $S'$. 

Euler equations (Sod Shock Tube)

Approximate eigenvalues \((v_{LR}^{Roe,*} - c_{LR}^{Roe,*})(\xi, \eta)_{\eta=0,...,N_o} \) (red), \((v_{LR}^{Roe,*})(\xi, \eta)_{\eta=0,...,N_o} \) (green), and \((v_{LR}^{Roe,*} + c_{LR}^{Roe,*})(\xi, \eta)_{\eta=0,...,N_o} \) (blue) corresponding to each stochastic element together with their density functions. Computations at \(t = 1\) with \(N_r = 3\) and \(N_o = 2\).
**CPU improvements**

- Only the eigenvalue \( \nu - c \) can change its sign.
- **Mean value averaged criterium** → portions of \((x, \xi)\) actually selected for the entropy correction:
  \[ x_{L-1/2}, x_{R+1/2}[\times \alpha_\sigma \text{ such that } E^{\alpha_\sigma}[v^{\text{Roe},*}_{LR} - c^{\text{Roe},*}_{LR}] \text{ changes its sign at the physical interface } LR. \]
- Use a numerical tolerance \( c_{tol} (= C_{\nu_{ref}}) \)

\[
E^{\alpha_\sigma}[(v^*_L - c^*_L)] - c_{tol} < 0 < E^{\alpha_\sigma}[(v^*_R - c^*_R)] + c_{tol}.
\]
Euler equations (Sod Shock Tube)  

Portion of the domain \((x, \xi)\) selected for the entropy correction and card \(S'\). Value\(=-1\) if no correction. Value\(=0\) if test. Card \(S'\) else. Computations with \(ctol = 1 e^{-2}\) (left) and \(ctol = 1 e^{-5}\) (right).

<table>
<thead>
<tr>
<th>(\dim S^{Nr,No})</th>
<th>No = 1, Nr = 3</th>
<th>No = 2, Nr = 3</th>
<th>No = 3, Nr = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ctol = +\infty)</td>
<td>(T_{CPU}) 11.7</td>
<td>(T_{CPU}) 16.1</td>
<td>(T_{CPU}) 21.6</td>
</tr>
<tr>
<td>(ctol = 1 e^{-1})</td>
<td>(factor) 1.0e-0</td>
<td>(factor) 1.0e-0</td>
<td>(factor) 1.0e-0</td>
</tr>
<tr>
<td>(ctol = 1 e^{-2})</td>
<td>8.2</td>
<td>11.8</td>
<td>16.4</td>
</tr>
<tr>
<td>(ctol = 1 e^{-3})</td>
<td>6.5</td>
<td>9.8</td>
<td>13.9</td>
</tr>
<tr>
<td>(ctol = 0) (\epsilon_h)</td>
<td>6.1</td>
<td>9.3</td>
<td>13.5</td>
</tr>
<tr>
<td></td>
<td>1.32e-3</td>
<td>2.88e-4</td>
<td>7.17e-4</td>
</tr>
</tbody>
</table>

| \(ctol = 1 e^{-2}\) | 9.2 | 2.8e-3 | 2.5e-3 |
| \(ctol = 1 e^{-3}\) | 9.3 | 2.8e-3 | 2.5e-3 |
| \(ctol = 0\) \(\epsilon_h\) | 9.2 | 2.8e-3 | 2.5e-3 |
Conclusion

- fully intrusive multi-resolution scheme
- Roe-type solver with upwind matrices efficiently computed by an original and fast method
- accurate and robust method
- entropy correction in the presence of sonic points only requiring marginal costs
- yet, computational costs scale as $\dim S^{\text{No}, \text{Nr}}$ (at least for moderate No)
- savings in computational costs for problems with higher stochastic dimensions → adaptive stochastic mesh refinement
References:

Intrusive Projection Methods with Upwinding for Uncertain Nonlinear Hyperbolic Systems (submitted)
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern

Roe solver with Entropy Corrector for Uncertain Hyperbolic Systems (submitted)
J. Tryoen, O. Le Maître, M. Ndjinga, A. Ern
Case of random variables with non-uniform distribution functions

Stochastic parametrization

\( \xi = (\xi_1, \ldots, \xi_N) \) vector of random variables with known independent distribution functions.

Change of variables

\( x(\xi) = (x_1(\xi_1), \ldots, x_N(\xi_N)) = (p_1(\xi_1), \ldots, p_N(\xi_N)) \) with 
\( (p_d(\xi_d))_{d=1,\ldots,N} \) cumulative density functions 
\[ \rightarrow x(\xi) \sim U([0, 1]^N). \]

Expansion of a process

\[ H(\xi) = \tilde{H}(x(\xi)) = \sum_{\alpha=1}^{P} \tilde{H}_\alpha \psi_\alpha(x(\xi)). \]
Burgers equation

\[
\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad F(U) = \frac{U^2}{2},
\]

\[
U^+(\xi_1) = 1 + 0.1(2\xi_1 - 1), \quad \xi_1 \sim \mathcal{U}[0, 1],
\]

\[
U^-(\xi_2) = -1 + 0.05(2\xi_2 - 1), \quad \xi_2 \sim \mathcal{U}[0, 1].
\]
Error $L^2$ on the eigenvalues of $|\nabla u f(u_{LR}^{Roe})|$ at $t = 0.4$. 

Error $L^\infty$ on the eigenvalues of $|\nabla u f(u_{LR}^{Roe})|$ at $t = 0.4$.

Stochastic error at $t = 0.6$ on the semi-discrete solution.
Stochastic error $\epsilon_h(x, t)$ for early (left) and longer (right) times.

Stochastic error $\epsilon_h(x, t = 6.5)$ for various $\text{No}$ and $\text{Nr}$. Computations with $\text{Nc} = 250$. 

Euler equations