

EXACT RECONSTRUCTION USING BEURLING MINIMAL EXTRAPOLATION

YOHANN DE CASTRO AND FABRICE GAMBOA

ABSTRACT. We show that measures with finite support on the real line are the unique solution to an algorithm, named generalized minimal extrapolation, involving only a finite number of generalized moments (which encompass the standard moments, the Laplace transform, the Stieltjes transformation, etc.).

Generalized minimal extrapolation shares related geometric properties with the basis pursuit approach of Chen, Donoho and Saunders [CDS98]. Indeed we also extend some standard results of compressed sensing (the dual polynomial, the nullspace property) to the signed measure framework.

We express exact reconstruction in terms of a simple interpolation problem. We prove that every nonnegative measure, supported by a set containing s points, can be exactly recovered from only $2s + 1$ generalized moments. This result leads to a new construction of deterministic sensing matrices for compressed sensing.

INTRODUCTION

In the last decade much emphasis has been put on the exact reconstruction of sparse finite dimensional vectors using the *basis pursuit* algorithm. The pioneering paper of Chen, Donoho and Saunders [CDS01] has brought this method to the statistics community. Note that the seminal ideas on the subject appeared in earlier works of Donoho and Stark [DS89]. Therein, mainly the discrete Fourier transform is considered. Similarly, P. Doukhan, E. Gassiat and one author of this present paper [DG96, GG96] considered the exact reconstruction of a nonnegative measure. More precisely, they derived results when one only knows the values of a finite number of linear functionals at the target measure. Moreover, they study stability with respect to a metric for weak convergence which is not the case here.

In this paper, we are concerned with the measure framework. We show that the exact reconstruction of a **signed** measure is still possible when one knows the values of only a finite number of non-adaptive linear measurements. Surprisingly our method, called *generalized minimal extrapolation*, appears to uncover exact reconstruction results related to basis pursuit.

Let us explain more precisely what is done here. Consider a *signed discrete measure* σ on a set I . Unless otherwise specified, assume that $I := [-1, 1]$. Note that all our results easily extend to any real bounded set. Consider the *Jordan decomposition*,

$$\sigma = \sigma^+ - \sigma^-,$$

Key words and phrases. Beurling Minimal Extrapolation, Basis Pursuit, Compressed Sensing, Convex optimization.

and denote by \mathcal{S}^+ (resp. \mathcal{S}^-) the support of σ^+ (resp. σ^-). Let us define the *Jordan support* of the measure σ as the pair $\mathcal{J} := (\mathcal{S}^+, \mathcal{S}^-)$. Assume further that $\mathcal{S} := \mathcal{S}^+ \cup \mathcal{S}^-$ is **finite** and has cardinality s . Moreover suppose that \mathcal{J} belongs to a family \mathcal{Y} of pairs of subsets of I (see Definition 1 for more details). We call \mathcal{Y} a *Jordan support family*. The measure σ can be written as

$$\sigma = \sum_{i=1}^s \sigma_i \delta_{x_i},$$

where $\mathcal{S} = \{x_1, \dots, x_s\}$, $\sigma_1, \dots, \sigma_s$ are nonzero real numbers, and δ_x denotes the Dirac measure at point x .

Let $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ be **any** family of continuous functions on \bar{I} , where the set \bar{I} denotes the closure of I (this statement is meant to be general and encompasses the case where I is not closed). Let μ be a signed measure on I . The k -th *generalized moment* of μ is defined by

$$(1) \quad c_k(\mu) = \int_I u_k \, d\mu$$

for all the indices $k = 0, 1, \dots, n$.

Our main issue. We are concerned with the reconstruction of the *target measure* σ from the *observation* of $\mathcal{K}_n := (c_0(\sigma), \dots, c_n(\sigma))$, i.e. its first $(n+1)$ generalized moments. We assume that both the support \mathcal{S} and the weights σ_i of the target measure σ are **unknown**. We investigate if it is possible to recover σ uniquely from the observation of \mathcal{K}_n . More precisely, *does an algorithm fitting $\mathcal{K}_n(\sigma)$ among all the signed measures of I recover the measure σ ?*

Note that a finite number of assigned standard moments does not define a unique signed measure. In fact one can check that for each signed measure μ and for each integer $m \geq 1$ there exists a measure $\mu' \neq \mu$ having the same first m moments. It seems there is no hope of recovering discrete measures from a finite number of its generalized moments. Surprisingly, we show that every *extrema Jordan type* measure σ (see Definition 1 and the examples that follow) is the unique solution of a *total variation* minimizing algorithm, *generalized minimal extrapolation*.

Basis pursuit. In [CDS98] Chen, Donoho and Saunders introduced *basis pursuit*. It is the process of reconstructing a target vector $x_0 \in \mathbb{R}^p$ from the observation $b = Ax_0$ by finding a sparse solution x^* to an under-determined system of equations:

$$(BP) \quad x^* \in \underset{y \in \mathbb{R}^p}{\text{Arg min}} \|y\|_1 \quad \text{s.t. } Ay = Ax_0,$$

where $A \in \mathbb{R}^{n \times p}$ is the *design matrix*. This program is one of the first steps toward a remarkable theory called *compressed sensing* [CRT06a, Don06]. As a result, this extremum is appropriated to the reconstruction of *sparse* vectors (i.e. vectors with a small support [Don06]). In this paper we develop a related program that recovers all the measures with enough structured Jordan support (which can be seen as the sparsity-related measures).

Generalized minimal extrapolation. Denote by \mathcal{M} the set of finite signed measures on I and by $\|\cdot\|_{TV}$ the *total variation* norm. We recall that for all $\mu \in \mathcal{M}$,

$$\|\mu\|_{TV} = \sup_{\Pi} \sum_{E \in \Pi} |\mu(E)|,$$

where the supremum is taken over all partitions Π of I into a finite number of disjoint measurable subsets. By analogy with basis pursuit, *generalized minimal extrapolation* is the process of reconstructing a target measure σ from the observation $\mathcal{K}_n(\sigma) = (c_0(\sigma), \dots, c_n(\sigma))$ of its first $n + 1$ generalized moments $c_k(\sigma)$ by finding a solution of the problem

$$(GME) \quad \sigma^* \in \underset{\mu \in \mathcal{M}}{\text{Arg min}} \|\mu\|_{TV} \quad \text{s.t. } \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma).$$

On one hand, basis pursuit minimizes the ℓ_1 -norm subject to linear constraints. On the other hand, generalized minimal extrapolation naturally substitutes the TV-norm (the total variation norm) for the ℓ_1 -norm. For the case of Fourier coefficients, (GME) is simply *Beurling Minimal Extrapolation* [Beu38]. The program (GME) is named after this remark.

Let us emphasize that generalized minimal extrapolation looks for a minimizer among **all signed** measures on I . Nevertheless, the target measure σ is assumed to be of *extrema Jordan type*.

Extrema Jordan type measures. Let us define more precisely what we understand by the Jordan support family Y .

Definition 1 (Extrema Jordan type measure) — *We say that a signed measure μ is of extrema Jordan type (with respect to a family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$) if and only if its Jordan decomposition $\mu = \mu^+ - \mu^-$ satisfies*

$$\text{Supp}(\mu^+) \subset E_p^+ \quad \text{and} \quad \text{Supp}(\mu^-) \subset E_p^-,$$

where $\text{Supp}(v)$ is defined as the support of the measure v , and

- P denotes any linear combination of elements of \mathcal{F} ,
- P is not constant and $\|P\|_\infty \leq 1$,
- E_p^+ (resp. E_p^-) is the set of all points x_i such that $P(x_i) = 1$ (resp. $P(x_i) = -1$).

In the following, we give some examples of extrema Jordan type measures with respect to the family

$$\mathcal{F}_p^n = \{1, x, x^2, \dots, x^n\}.$$

These measures can be seen as "interesting" target measures for (GME) given observation of the first $n + 1$ standard moments.

Examples with respect to the family \mathcal{F}_p^n . For the sake of readability, let $n = 2m$ be an even integer. We present three important examples.

Nonnegative measures: The nonnegative measures whose support has size s not greater than $n/2$ are extrema Jordan type measures. Indeed, let σ be a nonnegative measure and $\mathcal{S} = \{x_1, \dots, x_s\}$ be its support. Set

$$P = 1 - c \prod_{i=1}^s (x - x_i)^2.$$

Then, for a sufficiently small value of the parameter c , the polynomial P has supremum norm not greater than 1. The existence of such a polynomial shows that the measure σ is an extrema Jordan type measure.

In Section 2 we extend this notion to any *homogeneous M-system*.

Chebyshev measures: The k -th *Chebyshev polynomial of the first order* is defined by

$$(2) \quad T_k(x) = \cos(k \arccos(x)), \quad \forall x \in [-1, 1].$$

It is well known that it has supremum norm not greater than 1, and that

$$\bullet \ E_{T_k}^+ = \left\{ \cos(2l\pi/k), \ l = 0, \dots, \lfloor \frac{k}{2} \rfloor \right\},$$

$$\bullet \ E_{T_k}^- = \left\{ \cos((2l+1)\pi/k), \ l = 0, \dots, \lfloor \frac{k}{2} \rfloor \right\},$$

whenever $k > 0$. Then, any measure σ such that

$$\text{Supp}(\sigma^+) \subset E_{T_k}^+ \quad \text{and} \quad \text{Supp}(\sigma^-) \subset E_{T_k}^-,$$

for some $0 < k \leq n$, is an extrema Jordan type measure.

Further examples are presented in Section 3.

Δ -spaced out type measures: Let Δ be a positive real and S_Δ be the set of all pairs (S^+, S^-) of subsets of $[-1, 1]$ such that

$$\forall x, y \in S^+ \cup S^-, \ x \neq y, \quad |x - y| \geq \Delta.$$

In Lemma 4.2, we prove that, for all $(S^+, S^-) \in S_\Delta$, there exists a polynomial $P_{(S^+, S^-)}$ such that

- $P_{(S^+, S^-)}$ has degree n not greater than a bound depending only on Δ ,
- $P_{(S^+, S^-)}$ is equal to 1 on the set S^+ ,
- $P_{(S^+, S^-)}$ is equal to -1 on the set S^- ,
- and $\|P_{(S^+, S^-)}\|_\infty \leq 1$.

This shows that any measure σ with Jordan support included in S_Δ is an extrema Jordan type measure.

In this paper, we give exact reconstruction results for these three kinds of extrema Jordan type measures. In fact, our results extend to others families \mathcal{F} . Roughly, they can be stated as follows:

Nonnegative measures: Assume that \mathcal{F} is a homogeneous M -system (see 2.1.3). Theorem 2.1 shows that *any nonnegative measure σ is the unique solution of generalized minimal extrapolation given the observation $\mathcal{K}_n(\sigma)$, where n is not less than twice the size of the support of σ .*

Generalized Chebyshev measures: Assume that \mathcal{F} is an M -system (see definition 2.1.2). Proposition 3.3 shows the following result: *Let σ be a signed measure having Jordan support included in $(E_{\mathfrak{T}_k}^+, E_{\mathfrak{T}_k}^-)$, for some $1 \leq k \leq n$, where \mathfrak{T}_k denotes the k -th generalized Chebyshev polynomial (see 3.3.1). Then σ is the unique solution to generalized minimal extrapolation (GME) given $\mathcal{K}_n(\sigma)$, i.e. its first $(n+1)$ generalized moments.*

Δ -interpolation: Considering the standard family $\mathcal{F}_p^n = \{1, x, x^2, \dots, x^n\}$, Proposition 4.3 shows that *generalized minimal extrapolation exactly recovers any Δ -spaced out type measure σ from the observation $\mathcal{K}_n(\sigma)$, where n is greater than a bound depending only on Δ .*

These results are closely related to standard results of basis pursuit [Don06]. In fact, further analogies with compressed sensing can be emphasized.

Analogy with compressed sensing. Our estimator follows the aura of the recent breakthroughs [CDS98, CRT06a] in compressed sensing.

In the past decade E. J. Candès, J. Romberg, and T. Tao have shown [CRT06b] that it is possible to exactly recover all sparse vectors from few linear measurements. They considered a matrix $A \in \mathbb{R}^{n \times p}$ with i.i.d entries (centered Gaussian, Bernoulli, random Fourier sampling) and an s -sparse vector x_0 (i.e. vector with support of size at most s). They pointed out that, with very high probability, the vector x_0 is the only point of contact between the ℓ_1 -ball of radius $\|x_0\|_1$ and the affine space $\{y, Ay = Ax_0\}$. This result holds as soon as $n \geq C s \log(p/s)$, where $C > 0$ is a universal constant. In our framework we uncover the same geometric property:

Let σ be an extrema Jordan type measure. Then σ is a point of contact between the ball of radius $\|\sigma\|_{TV}$ and the affine space $\{\mu \in \mathcal{M}, \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\}$, where n is greater than a bound depending only on the structure of the Jordan support of σ . For instance, in the nonnegative measure case, if σ has support of size at most s , then $n = 2s$ suffices (see Theorem 2.1).

Actually the reader can check that the above property is equivalent to the fact that the measure σ is a solution of generalized minimal extrapolation (more details can be found in Section 1.2). Accordingly, generalized minimal extrapolation (GME) minimizes the total variation in order to pursue support of the target measure.

Organization. This paper falls into four parts. The next section introduces *generalized dual polynomials* and shows that exact recovery can be understood in terms of an interpolation problem. Section 2 studies the exact reconstruction of non-negative measures, and gives *explicit* construction of design matrices for basis pursuit. Section 3 focuses on generalized Chebyshev polynomials and shows that it is possible to reconstruct signed measures from very few generalized moments. The last section uncovers a property related to the nullspace property of compressed sensing.

1. GENERALIZED DUAL POLYNOMIALS

In this section we introduce generalized dual polynomial. In particular we are concerned with a sufficient condition that guarantees the exact reconstruction of the measure σ . In fact, this condition relies on an interpolation problem.

1.1. An interpolation problem. An insight into exact reconstruction is given by Lemma 1.1. Roughly, the existence of a generalized dual polynomial is a sufficient condition for the exact reconstruction of a signed measure with finite support.

As usual, the following result holds for any family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ of continuous functions on \bar{I} . Throughout, $\text{sgn}(x)$ denotes the sign of the real x .

Lemma 1.1 (The generalized dual polynomials) — *Let n be a positive integer. Let $\mathcal{S} = \{x_1, \dots, x_s\} \subset I$ be a subset of size s and $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$. If there exists a linear combination $P = \sum_{k=0}^n a_k u_k$ such that*

(i) the generalized Vandermonde system

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & \dots & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix}$$

has full column rank,

(ii) $P(x_i) = \varepsilon_i, \forall i = 1, \dots, s,$

(iii) $|P(x)| < 1, \forall x \in [-1, 1] \setminus \mathcal{S},$

Then every measure $\sigma = \sum_{i=1}^s \sigma_i \delta_{x_i}$, such that $\text{sgn}(\sigma_i) = \varepsilon_i$, is the **unique** solution of generalized minimal extrapolation given the observation $\mathcal{K}_n(\sigma)$.

Proof. See A.1. □

The linear combination P considered in the Lemma 1.1 is called a **generalized dual polynomial**. This naming is inherited from the original article [CRT06a] of Candès, Tao and Romberg, and the *dual certificate* named by Candès and Plan [CP10].

1.2. Reconstruction of a cone. Given a subset $\mathcal{S} = \{x_1, \dots, x_s\}$ and a sign sequence $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$, Lemma 1.1 shows that if the generalized interpolation problem defined by (i), (ii) and (iii) has a solution then generalized minimal extrapolation recovers exactly all measures σ with support \mathcal{S} and such that $\text{sgn}(\sigma_i) = \varepsilon_i$.

Let us emphasize that the result is slightly stronger. Indeed the proof of A.1 remains unchanged if some coefficients σ_i are zero. Consequently (GME) recovers exactly all the measures σ of which support is **included** in $\mathcal{S} = \{x_1, \dots, x_s\}$ and such that $\text{sgn}(\sigma_i) = \varepsilon_i$ for all nonzero σ_i .

Let us denote this set by $\mathcal{C}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$. It is exactly the *cone* defined by

$$\mathcal{C}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s) = \left\{ \sum_{i=1}^s \mu_i \delta_{x_i} \mid \forall \mu_i \neq 0, \text{sgn}(\mu_i) = \varepsilon_i \right\}.$$

Thus the existence of P implies the exact reconstruction of **all** measures in this cone. The cone $\mathcal{C}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$ is the conic span of an $(s-1)$ -dimensional face of the TV-unit ball, that is

$$\mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s) = \left\{ \sum_{i=1}^s \varepsilon_i \lambda_i \delta_{x_i} \mid \forall i, \lambda_i \geq 0 \text{ and } \sum_{i=1}^s \lambda_i = 1 \right\}.$$

Furthermore, the affine space $\{\mu, \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\}$ is tangent to the TV-unit ball at any point $\sigma \in \mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$, as shown in the following remark.

Remark. From a convex optimization point of view, the *dual certificates* [CP10] and the generalized dual polynomials are deeply related: the existence of a generalized dual polynomial P implies that, for all $\sigma \in \mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$, a subgradient Φ_P of the TV-norm at the point σ is perpendicular to the set of the feasible points, that is

$$\{\mu, \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\} \subset \ker(\Phi_P),$$

where \ker denotes the nullspace. A proof of this remark can be found in A.2.

1.3. On condition (i) in Lemma 1.1. Obviously, when $u_k = x^k$ for $k = 0, 1, \dots, n$, conditions (ii) and (iii) imply that $n \geq s$ and so condition (i). Nevertheless, this implication is not true for a general set of functions $\{u_0, u_1, \dots, u_n\}$. Moreover, Lemma 1.1 can fail if condition (i) is not satisfied. For example, set $n = 0$ and consider a continuous function u_0 satisfying the two conditions (ii) and (iii). In this case, if the target σ belongs to $\mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$ (where x_1, \dots, x_s and $\varepsilon_1, \dots, \varepsilon_s$ are given by (ii) and (iii)), then **every** measure $\mu \in \mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$ is a solution of generalized minimal extrapolation given the observation $\mathcal{K}_0(\sigma)$. Indeed,

$$\|\mu\|_{TV} = \int_{-1}^1 u_0 \, d\mu = \mathcal{K}_0(\mu),$$

for all $\mu \in \mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$. This example shows that condition (i) is necessary. Reading the proof A.1, conditions (ii) and (iii) ensure that the solutions to generalized minimal extrapolation belong to the cone $\mathcal{C}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$, whereas condition (i) gives uniqueness.

1.4. The extrema Jordan type measures. Lemma 1.1 shows that Definition 1 is well-founded. In fact, we have the the following corollary.

Corollary — *Let σ be an extrema Jordan type measure. Then the measure σ is a solution to generalized minimal extrapolation given the observation $\mathcal{K}_n(\sigma)$.*

*Furthermore, if the Vandermonde system given by (i) in Lemma 1.1 has full column rank (where $\mathcal{S} = \{x_1, \dots, x_s\}$ denotes the support of σ), then the measure σ is the **unique** solution to generalized minimal extrapolation given the observation $\mathcal{K}_n(\sigma)$.*

This corollary shows that the "extrema Jordan type" notion is appropriate to exact reconstruction using generalized minimal extrapolation.

2. EXACT RECONSTRUCTION OF THE NONNEGATIVE MEASURES

In this section we show that if the underlying family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a *homogeneous M-system* then (GME) recovers exactly each finitely supported nonnegative measure μ from the observation of a surprisingly few generalized moments. We begin with the definition of homogeneous M-systems.

2.1. Markov systems. Markov systems were introduced in approximation theory [KN77, BE95, KS66]. They deal with the problem of finding the best approximation, in terms of the ℓ_∞ -norm, of a given continuous function in ℓ_∞ norm. We begin with the definition of Chebyshev systems (the so-called T-system). They can be seen as a natural extension of algebraic monomials. Thus a finite combination of elements of a T-system is called a *generalized polynomial*.

2.1.1. T-systems of order k. Denote by $\{u_0, u_1, \dots, u_k\}$ a set of continuous real (or complex) functions on \bar{I} . This set is a T-system of degree k if and only if every generalized polynomial

$$P = \sum_{l=0}^k a_l u_l,$$

where $(a_0, \dots, a_k) \neq (0, \dots, 0)$, has at most k zeros in I .

This definition is equivalent to each of the two following conditions:

- For all x_0, x_1, \dots, x_k distinct elements of I and all y_0, y_1, \dots, y_k real (or complex) numbers, there exists a unique generalized polynomial P (i.e. $P \in \text{Span}\{u_0, u_1, \dots, u_k\}$) such that $P(x_i) = y_i$, for all $i = 0, 1, \dots, k$.
- For all x_0, \dots, x_k distinct elements of I , *generalized Vandermonde system*

$$\begin{pmatrix} u_0(x_0) & u_0(x_1) & \dots & u_0(x_k) \\ u_1(x_0) & u_1(x_1) & \dots & u_1(x_k) \\ \vdots & \vdots & \dots & \vdots \\ u_k(x_0) & u_k(x_1) & \dots & u_k(x_k) \end{pmatrix}$$

has full rank.

2.1.2. M -systems. We say that the family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is an M -system if and only if it is a T -system of degree k for all $0 \leq k \leq n$. Actually, M -systems are common objects (see [KN77]). We mention some examples below.

In this paper, we are concerned with target measures on $I = [-1, 1]$. Usually M -systems are defined on general Hausdorff spaces (see [BEZ94] for instance). For the sake of readability, we present examples with different values of I . In each case, our results easily extend to target measures with finite support included in the corresponding I . As usual, if not specified, the set I is assumed to be $[-1, 1]$.

Real polynomials: The family $\mathcal{F}_p = \{1, x, x^2, \dots\}$ is an M -system. The real polynomials give the standard moments.

Müntz polynomials: Let $0 < \alpha_1 < \alpha_2 < \dots$ be any real numbers. The family $\mathcal{F}_m = \{1, x^{\alpha_1}, x^{\alpha_2}, \dots\}$ is an M -system on $I = [0, +\infty)$.

Trigonometric functions: The family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ is an M -system on $I = [0, 1]$.

Characteristic function: The family $\mathcal{F}_c = \{1, \exp(i\pi x), \exp(i2\pi x), \dots\}$ is an M -system on $I = [-1, 1]$. The moments are the *characteristic function* of σ at points $k\pi$, $k \in \mathbb{N}$. It yields

$$c_k(\sigma) = \int_{-1}^1 \exp(ik\pi t) d\sigma(t) = \varphi_\sigma(k\pi).$$

In this case, the underlying scalar field is \mathbb{C} .

Stieltjes transformation: The family $\mathcal{F}_s = \{\frac{1}{z_1-x}, \frac{1}{z_2-x}, \dots\}$, where none of the z_k 's belongs to $[-1, 1]$, is an M -system. The corresponding moments are the *Stieltjes transformation* $S_\sigma(z_k)$ of σ , namely

$$c_k(\sigma) = \int_{-1}^1 \frac{d\sigma(t)}{z_k - t} = S_\sigma(z_k).$$

Laplace transform: The family $\mathcal{F}_l = \{1, \exp(-x), \exp(-2x), \dots\}$ is an M -system. The moments are the *Laplace transform* $\mathcal{L}\sigma$ at integer points, namely

$$c_k(\sigma) = \int_{-1}^1 \exp(-kt) d\sigma(t) = \mathcal{L}\sigma(k).$$

A broad variety of common families can be considered in our framework. The above list is not meant to be exhaustive.

Consider the family $\mathcal{F}_s = \{\frac{1}{z_0-x}, \frac{1}{z_1-x}, \dots\}$. Note that no linear combination of its elements gives the constant function 1. Thus the constant function 1 is

not a generalized polynomial of this system. To treat such cases, we introduce *homogeneous M-systems*.

2.1.3. Homogeneous M-systems. We say that a family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a *homogeneous M-system* if and only if it is an M-system and u_0 is a constant function. In this case, all constant functions c , with $c \in \mathbb{R}$ (or \mathbb{C}), are generalized polynomials. Hence the field \mathbb{R} (or \mathbb{C}) is naturally embedded in generalized polynomials. The adjective homogeneous is named after this comment.

From any M-system we can always construct a homogeneous M-system. Indeed, let $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ be an M-system. In particular the family \mathcal{F} is a T-system of order 0. Thus the continuous function u_0 does not vanish in $[-1, 1]$. In fact the family $\{1, \frac{u_1}{u_0}, \frac{u_2}{u_0}, \dots, \frac{u_n}{u_0}\}$ is a homogeneous M-system.

All the previous examples of M-systems (see 2.1.2) are homogeneous, even Stieltjes transformation:

$$\tilde{\mathcal{F}}_s = \left\{ 1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \dots \right\}.$$

Using homogeneous M-systems, we show that one can exactly recover **all** nonnegative measures from a few generalized moments.

2.2. An important theorem. The following result is one of the main theorems of our paper. It states that the generalized minimal extrapolation (**GME**) recovers **all nonnegative measures** σ whose support is of size s from **only** $2s + 1$ generalized moments.

Theorem 2.1 — *Let \mathcal{F} be an homogeneous M-system on I . Consider a nonnegative measure σ with finite support included in I . Then the measure σ is the **unique** solution to generalized minimal extrapolation given observation $\mathcal{K}_n(\sigma)$, where n is not less than twice the size of the support of σ .*

Proof. The complete proof can be found in B.1 but some key points from *the theory of approximation* are presented in 2.2.1. For further insights about *Markov systems*, we recommend the books [KN77, KS66]. \square

In addition, this result is sharp in the following sense. Every measure with support size s depends on $2s$ parameters (s for its support and s for its weights). Surprisingly, this information can be recovered from only $2s + 1$ of its generalized moments. Furthermore the program (**GME**) does not use the fact that the target is **nonnegative**. It recovers σ among **all signed** measures with finite support.

2.2.1. Nonnegative interpolation. An important property of M-systems is the existence of a nonnegative generalized polynomial that vanishes exactly at a prescribed set of points $\{t_1, \dots, t_m\}$, where $t_i \in I$ for all $i = 1, \dots, m$. Indeed, define the *index* as

$$(3) \quad \text{Index}(t_1, \dots, t_m) = \sum_{j=1}^m \chi(t_j),$$

where $\chi(t) = 2$ if t belongs to $\overset{\circ}{I}$ (the interior of I) and 1 otherwise. The next lemma guarantees the existence of nonnegative generalized polynomials.

Lemma 2.2 (Nonnegative generalized polynomial) — Consider an M -system \mathcal{F} and points t_1, \dots, t_m in I . These points are the **only** zeros of a nonnegative generalized polynomial of degree at most n **if and only if** $\text{Index}(t_1, \dots, t_m) \leq n$.

A proof of this lemma is in [KN77]. Note that this lemma holds for **all** M -systems. However our main theorem needs a *homogeneous* M -system.

2.2.2. *Is homogeneous necessary?* If one considers non-homogeneous M -systems then it is possible to give counterexamples that go against Theorem 2.1 for all $n \geq 2s$. Indeed, we have the next result.

Proposition 2.3 — Let σ be a nonnegative measure supported by s points. Let n be an integer such that $n \geq 2s$. Then there exists an M -system \mathcal{F} and a measure $\mu \in \mathcal{M}$ such that $\mathcal{K}_n(\sigma) = \mathcal{K}_n(\mu)$ and $\|\mu\|_{TV} < \|\sigma\|_{TV}$.

Proof. See B.2. □

Theorem 2.1 gives us the opportunity to build a large family of *deterministic* matrices for compressed sensing in the case of nonnegative signals.

2.3. **Deterministic matrices for compressed sensing.** The heart of this article lies in the next theorem. It gives **deterministic** matrices for *compressed sensing*. We begin with some *state-of-the-art* results in compressed sensing. In the following, p denotes the number of predictors (or, from a signal processing view point, the length of the signal).

Deterministic Design: As far as we know, for

$$n = \mathcal{O}_{p, s \rightarrow \infty} \left(s \log \left(\frac{p}{s} \right) \right),$$

there exists [BGI⁺08] a deterministic matrix $A \in \mathbb{R}^{n \times p}$ such that basis pursuit (BP) recovers all s -sparse vectors from the observation Ax_0 .

Random Design: If

$$n \geq C s \log \left(\frac{p}{s} \right),$$

where $C > 0$ is a universal constant, then there exists (with high probability) a random matrix $A \in \mathbb{R}^{n \times p}$ such that basis pursuit recovers all s -sparse vectors from the observation Ax_0 .

The deterministic result holds for *large* values of s , n and p . For readability we do not specify the sense of *large* here. The reader may find an abundant literature in the respective references (see for example [BGI⁺08, Don06]).

Considering nonnegative sparse vectors, it is possible to drop the bound on n to

$$n \geq 2s + 1.$$

Unlike the above examples, this result holds for *all* values of the parameters (as soon as $n \geq 2s + 1$). In addition it give **explicit** design matrices for basis pursuit. Last but not least, this bound on n does not depend on p . In special cases, this result has been previously developed in [DJHS92, Fuc96, DT05, DT10]. Using Theorem 2.1, it is possible to provide a generalization of this result to a broad range of measurement matrices.

Theorem 2.4 (Deterministic Design Matrices) — *Let n, p, s be integers such that*

$$s \leq \min(n/2, p).$$

Let $\{1, u_1, \dots, u_n\}$ be a homogeneous M -system on I . Let t_1, \dots, t_p be distinct reals of I . Let A be generalized Vandermonde system defined by

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ u_1(t_1) & u_1(t_2) & \dots & u_1(t_p) \\ u_2(t_1) & u_2(t_2) & \dots & u_2(t_p) \\ \vdots & \vdots & \dots & \vdots \\ u_n(t_1) & u_n(t_2) & \dots & u_n(t_p) \end{pmatrix}.$$

Then basis pursuit (BP) exactly recovers all nonnegative s -sparse vectors $x_0 \in \mathbb{R}^p$ from the observation Ax_0 .

Proof. See B.3. □

Remark. The purely analytical components of this result are traceable back to the theory of neighborly polytopes (see for instance [DT05]) and in some sense to the theory of moment problems which essentially follows from Carathéodory's work [Car07, Car11]. Other relevant work includes [KS53, Der56, Stu88]. This list is not meant to be exhaustive.

Although the predictors could be highly correlated, basis pursuit exactly recovers the target vector x_0 . Of course, this result is theoretical. In practice, the sensing matrix A can be very ill-conditioned. In this case, basis pursuit behaves poorly.

Numerical experiments. Our numerical experiments illustrate Theorem 2.4. They are of the following form:

- (a) Choose constants s (sparsity), n (number of known moments), and p (length of the vector). Choose the family \mathcal{F} (cosine, polynomial, Laplace, Stieltjes,...).
- (b) Select the subset \mathcal{S} (of size s) uniformly at random.
- (c) Randomly generate an s -sparse vector x_0 of support \mathcal{S} whose nonzero entries have the chi-square distribution with 1 degree of freedom.
- (d) Compute the observation Ax_0 .
- (e) Solve (BP), and compare with the target vector x_0 .

The program (BP) can be recast as a linear program (see [CDS01] for instance). Then we use an interior point method to solve (BP).

The entries of the target signal are distributed according to chi-square distribution with 1 degree of freedom. We chose this distribution to ensure that the entries are nonnegative. Let us emphasize that the actual values of x_0 can be arbitrary; only the sign matters. The result remains the same if we take the nonzero entries to be 1, say.

Let us denote $K : t \mapsto (1, u_1(t), \dots, u_n(t))$. The columns of A are the values of this map at points t_1, \dots, t_p . For large p , the vectors $K(t_i)$ can be highly correlated. In fact, the matrix A can be *ill-conditioned*. To avoid such a case, we chose a family such that the map K has a large derivative. It appears that the *cosine family* gives very good numerical results (see Figure 1).

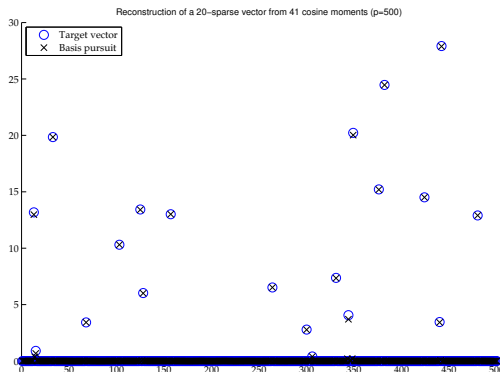


FIGURE 1. Consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ on $I = [0, 1]$ and the points $t_k = k/501$, for $k = 1, \dots, 500$. The blue circles represent the target vector x_0 (a 20-sparse vector), while the black crosses represent the solution x^* of (BP) from the observation of 41 cosine moments. In this example $s = 20$, $n = 41$, and $p = 500$. More numerical results can be found in Appendix D. This example shows that the reconstruction is excellent.

We investigate the reconstruction error between the numerical result \tilde{x} of the program (BP) and the target vector x_0 . Our experiment is of the following form:

- Choose p (length of the vector) and N (number of numerical experiments).
- Let s satisfy $1 \leq s \leq (p-1)/2$.
- Set $n = 2s + 1$ and solve the program (BP). Let \tilde{x} be the numerical result.
- Compute the ℓ_1 -error $\|\tilde{x} - x_0\|_1 / p$.
- Repeat N times the steps (c) and (d), and compute Err_s , the arithmetic mean of the ℓ_1 -errors.
- Return $\|\text{Err}_s\|_\infty$, the maximal value of Err_s .

For $p = 100$ and $N = 10$, we find that

$$\|\text{Err}_s\|_\infty \leq 0.05.$$

Note that all experiments were done for $n = 2s + 1$. This is the smallest value of n such that Theorem 2.3 holds.

3. EXACT RECONSTRUCTION FOR GENERALIZED CHEBYSHEV MEASURES

In this section we give some examples of extremal polynomials P as they appear in Definition 1. Considering M -systems, corollary of Lemma 1.1 shows that every measure with Jordan support included in (E_P^+, E_P^-) is the **only** solution to (GME). Indeed, condition (i) of Lemma 1.1 is clearly satisfied when the underlying family \mathcal{F} is an M -system.

3.1. Trigonometric families. In the context of M -systems we can exhibit some very particular dual polynomials. The global extrema of these polynomials gives families of support for which results of Lemma 1.1 hold.

The cosine family. First, consider the $(n + 1)$ -dimensional cosine system

$$\mathcal{F}_{\cos}^n := \{1, \cos(\pi x), \dots, \cos(n\pi x)\}$$

on $I = [0, 1]$. Obviously, extremal polynomials

$$P_k(x) = \cos(k\pi x),$$

for $k = 1, \dots, n$, satisfy $\|P_k\|_{\infty} \leq 1$ and $P_k(l/k) = (-1)^l$, for $l = 0, 1, \dots, (k - 1)$. According to Definition 1, let us denote

- $E_{P_k}^+ := \{2l/k \mid l = 0, \dots, \lfloor \frac{k-1}{2} \rfloor\}$,
- $E_{P_k}^- := \{(2l - 1)/k \mid l = 1, \dots, \lfloor \frac{k}{2} \rfloor\}$.

The corollary that follows Lemma 1.1 asserts the following result.

Consider a signed measure σ having Jordan support (S^+, S^-) such that $S^+ \subset E_{P_k}^+$ and $S^- \subset E_{P_k}^-$, for some $1 \leq k \leq n$. Then the measure σ can be exactly reconstructed from the observation of

$$(4) \quad \int_0^1 \cos(k\pi t) d\sigma(t), \quad k = 0, 1, \dots, n.$$

Moreover, since the family \mathcal{F}_{\cos}^n is an M -system, condition (i) in Lemma 1.1 is satisfied. Hence, the measure σ is the only solution of (GME) given the observations (4).

Using the classical mapping

$$\Psi : \begin{cases} [0, 1] & \rightarrow & [-1, 1] \\ x & \mapsto & \cos(\pi x) \end{cases} ,$$

the system of function $(1, \cos(\pi x), \dots, \cos(n\pi x))$ can be push-forward to the system of functions $(1, T_1(x), \dots, T_n(x))$, where $T_k(x)$ is the so-called *Chebyshev polynomial of the first kind* of order k , $k = 1, \dots, n$ (see 3.2).

The characteristic function. By the same token, consider the complex valued M -system defined by

$$\mathcal{F}_c^n = \{1, \exp(i\pi x), \dots, \exp(in\pi x)\}$$

on $I = [0, 2)$. In this case, one can check that

$$P_{\alpha, k}(t) = \cos(k\pi(t - \alpha)), \quad \forall t \in [0, 2),$$

where $\alpha \in \mathbb{R}$ and $0 \leq k \leq n/2$, is a generalized polynomial. Following the previous example, we set

- $E_{P_{\alpha, k}}^+ := \{\alpha + 2l/k \pmod{2} \mid l = 0, \dots, \lfloor \frac{k-1}{2} \rfloor\}$,
- $E_{P_{\alpha, k}}^- := \{\alpha + (2l - 1)/k \pmod{2} \mid l = 1, \dots, \lfloor \frac{k}{2} \rfloor\}$.

Hence Lemma 1.1 can be applied. It yields the following:

*Any signed measure having Jordan support included in $(E_{P_{\alpha, k}}^+, E_{P_{\alpha, k}}^-)$, for some $\alpha \in \mathbb{R}$ and $1 \leq k \leq n/2$, is the **unique** solution of (GME) given the observation*

$$\int_0^2 \exp(ik\pi t) d\sigma(t) = \varphi_{\sigma}(k\pi), \quad \forall k = 0, \dots, n,$$

where $\varphi_\sigma(k\pi)$ has been defined in the previous section (see 2.1.2).

Note that the study of basis pursuit with this kind of trigonometric moments has been considered in the pioneering work of Donoho and Stark [DS89].

3.2. Chebyshev polynomials. As mentioned in the introduction, the k -th Chebyshev polynomial of the first order is defined by

$$T_k(x) = \cos(k \arccos(x)), \quad \forall x \in [-1, 1].$$

We give some well known properties of Chebyshev polynomials. The k -th Chebyshev polynomial satisfies the *equioscillation property* on $[-1, 1]$. In fact, there exists $k + 1$ points $\zeta_i = \cos(\pi i/k)$ with $1 = \zeta_0 > \zeta_1 > \dots > \zeta_k = -1$ such that

$$T_k(\zeta_i) = (-1)^i \|T_k\|_\infty = (-1)^i,$$

where the supremum norm is taken over $[-1, 1]$. Moreover, the Chebyshev polynomial T_k satisfies the following *extremal property*.

Theorem 3.1 ([Riv90, BE95]) — *We have*

$$\min_{p \in \mathcal{P}_{k-1}^{\mathbb{C}}} \|x^k - p(x)\|_\infty = \|2^{1-k} T_k\|_\infty = 2^{1-k},$$

where $\mathcal{P}_{k-1}^{\mathbb{C}}$ denotes the set of complex polynomials of degree less than $k - 1$, and the supremum norm is taken over $[-1, 1]$. Moreover, the minimum is uniquely attained by $p(x) = x^k - 2^{1-k} T_k(x)$.

These two properties, namely the equioscillation property and the extremal property, will be useful to us when we define generalized Chebyshev polynomial.

Using Lemma 1.1 we uncover an exact reconstruction result. Consider the family

$$\mathcal{F}_p^n = \{1, x, x^2, \dots, x^n\}$$

on $I = [-1, 1]$. Set

- $E_{T_k}^+ = \{ \cos(2l\pi/k), l = 0, \dots, \lfloor \frac{k}{2} \rfloor \},$
- $E_{T_k}^- = \{ \cos((2l+1)\pi/k), l = 0, \dots, \lfloor \frac{k}{2} \rfloor \}.$

The following result holds:

*Consider a signed measure σ having Jordan support included in $(E_{T_k}^+, E_{T_k}^-)$, for some $1 \leq k \leq n$. Then the measure σ is the **only** solution to (GME) given its first $(n + 1)$ standard moments.*

Note that this result is restrictive in the location of the support points: they are not sparse in the usual sense, because they must be precisely located. Nevertheless, it can be extended to any M -systems with the help of *generalized Chebyshev polynomials*.

3.3. Generalized Chebyshev polynomials. Following [BE95], we define generalized Chebyshev polynomials as follows. Let $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ be an M -system on I .

3.3.1. *Definition.* The *generalized Chebyshev polynomial*

$$\mathfrak{T}_k := \mathfrak{T}_k\{u_0, u_1, \dots, u_n; I\},$$

where $1 \leq k \leq n$, is defined by the following three properties:

- \mathfrak{T}_k is a generalized polynomial of degree k , i.e. $\mathfrak{T}_k \in \text{Span}\{u_0, u_1, \dots, u_k\}$,
- there exists $x_0 < x_1 < \dots < x_k$ such that

$$(5) \quad \text{sgn}(\mathfrak{T}_k(x_{i+1})) = -\text{sgn}(\mathfrak{T}_k(x_i)) = \pm \|\mathfrak{T}_k\|_\infty,$$

for $i = 0, 1, \dots, k-1$,

- and

$$(6) \quad \|\mathfrak{T}_k\|_\infty = 1 \quad \text{with} \quad \mathfrak{T}_k(\max I) > 0.$$

The existence and the uniqueness of such \mathfrak{T}_k is proved in [BE95]. Moreover, the following theorem shows that the extremal property implies the equioscillation property (5).

Theorem 3.2 ([Riv90, BE95]) — *The k -th generalized Chebyshev polynomial \mathfrak{T}_k exists and can be written as*

$$\mathfrak{T}_k = c \left(u_k - \sum_{i=0}^{k-1} a_i u_i \right),$$

where $a_0, a_1, \dots, a_{k-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| u_k - \sum_{i=0}^{k-1} a_i u_i \right\|_\infty,$$

and the normalization constant $c \in \mathbb{R}$ can be chosen so that \mathfrak{T}_k satisfies property (6).

Generalized Chebyshev polynomials give a new family of extrema Jordan type measures (see Definition 1). The corresponding target measures are named *Chebyshev measures*.

3.3.2. *Exact reconstruction of Chebyshev measures.* Considering the equioscillation property (5), set

- $E_{\mathfrak{T}_k}^+$ as the set of the alternation point x_i such that $\text{sgn}(\mathfrak{T}_k(x_i)) = \|\mathfrak{T}_k\|_\infty$,
- $E_{\mathfrak{T}_k}^-$ as the set of the alternation point x_i such that $\text{sgn}(\mathfrak{T}_k(x_i)) = -\|\mathfrak{T}_k\|_\infty$.

A direct consequence of the last definition is the following proposition.

Proposition 3.3 — *Let σ be a signed measure having Jordan support included in $(E_{\mathfrak{T}_k}^+, E_{\mathfrak{T}_k}^-)$, for some $1 \leq k \leq n$. Then σ is the **unique** solution to generalized minimal extrapolation (GME) given $\mathcal{K}_n(\sigma)$, i.e. its first $(n+1)$ generalized moments.*

In the special case $k = n$, Proposition 3.3 shows that (GME) recovers all signed measures with Jordan support included in $(E_{\mathfrak{T}_n}^+, E_{\mathfrak{T}_n}^-)$ from the first $(n+1)$ generalized moments. Note that $E_{\mathfrak{T}_n}^+ \cup E_{\mathfrak{T}_n}^-$ has size n . Hence, this proposition shows that, among all signed measure on $[-1, 1]$, (GME) can recover a signed measure of support size n from only $(n+1)$ generalized moments. In fact, any measure with Jordan support included in $(E_{\mathfrak{T}_n}^+, E_{\mathfrak{T}_n}^-)$ can be uniquely defined by only $(n+1)$ generalized moments.

As far as we know, it is difficult to give the corresponding generalized Chebyshev polynomials for a given family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$. Nevertheless, Borwein, Erdélyi, and Zhang [BEZ94] give the explicit form of \mathfrak{T}_k for rational spaces (i.e. the Stieltjes transformation in our framework). See also [DS89, HSS96] for some applications in *optimal design*.

3.3.3. *Construction of Chebyshev polynomials for Stieltjes transformation.* We consider the Stieltjes transformation described in Section 2. In this case, Chebyshev polynomials \mathfrak{T}_k can be precisely described. Consider homogeneous M -system on $[-1, 1]$ defined by

$$\tilde{\mathcal{F}}_s^n = \left\{ 1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \dots, \frac{1}{z_n - x} \right\},$$

where $(z_i)_{i=1}^k \subset \mathbb{C} \setminus [-1, 1]$.

Reproducing [BE95], we can construct generalized Chebyshev polynomials of the first kind. It yields

$$\mathfrak{T}_k(x) = \frac{1}{2}(f_k(z) + f_k(z)^{-1}), \quad \forall x \in [-1, 1],$$

where z is uniquely defined by $x = \frac{1}{2}(z + z^{-1})$ and $|z| < 1$, and f_k is a known analytic function in a neighborhood of the closed unit disk. Moreover this analytic function can be expressed in terms of only $(z_i)_{i=1}^k$. See [BE95] for further details.

4. THE NULLSPACE PROPERTY FOR MEASURES

In this section we consider **any** countable family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ of *continuous functions* on I . In particular we do not assume that \mathcal{F} is a non-homogeneous M -system. We seek a sufficient condition for exact reconstruction of signed measures. More precisely, we are concerned with giving a related property to the *nullspace property* [CDD09] of compressed sensing.

Note that the solutions to program (GME) depend only on the first $(n + 1)$ elements of \mathcal{F} and on the target measure σ . We investigate the condition that the family \mathcal{F} must satisfy to ensure exact reconstruction. In the meantime, Cohen, Dahmen and DeVore introduced [CDD09] a relevant condition, the *nullspace property*. Their property binds the geometry of the nullspace of A and the *best k -term approximation* of the target x_0 given the observation Ax_0 . This well known property can be stated as follows.

4.1. The nullspace property in compressed sensing. Let $A \in \mathbb{R}^{n \times p}$ be a matrix. We say that A satisfies the nullspace property of order s *if and only if* for all nonzero vectors h in the nullspace of A , and all subsets of entries S of size s ,

$$\|h_S\|_1 < \|h_{S^c}\|_1,$$

where h_S denotes the vector whose i -th entry is h_i if $i \in S$ and 0 otherwise. It is now standard that basis pursuit (BP) exactly recovers all s -sparse vectors x_0 (i.e. vectors with at most s nonzero entries) *if and only if* the design matrix A satisfies the nullspace property of order s .

In this section, we show that the same property holds for generalized minimal extrapolation. According to the compressed sensing literature, we keep the same name for this related property.

4.2. The nullspace property for generalized minimal extrapolation. Consider the linear map $\mathcal{K}_n : \mu \mapsto (c_0(\mu), \dots, c_n(\mu))$ from \mathcal{M} to \mathbb{R}^{n+1} . We refer to this map as the *generalized moment morphism*. Its nullspace $\ker(\mathcal{K}_n)$ is a linear subspace of \mathcal{M} . The *Lebesgue decomposition theorem* is the precious tool used to define the nullspace property.

4.2.1. *The S-atomic part.* Let $\mu \in \mathcal{M}$ and $S = \{x_1, \dots, x_s\}$ be a finite subset of I . Define $\Delta_S = \sum_{i=1}^s \delta_{x_i}$ as the *Dirac comb* with support S . The Lebesgue decomposition of μ with respect to Δ_S gives

$$(7) \quad \mu = \mu_S + \mu_{S^c},$$

where μ_S is a **discrete** measure whose support is included in S , and μ_{S^c} is a measure whose support is included in $S^c := I \setminus S$.

4.2.2. *The nullspace property with respect to a Jordan support family.* First, as in the standard compressed sensing context [CDD09], we define the nullspace property with respect to a Jordan support family Y . This property is only a sufficient condition for exact reconstruction of finite measure; see Proposition 4.1.

Definition 1 (Nullspace property with respect to a Jordan support family Y) — We say that the generalized moment morphism \mathcal{K}_n satisfies the nullspace property with respect to a Jordan support family Y **if and only if** it satisfies the following property. For all nonzero measures μ in the nullspace of \mathcal{K}_n , and for all $(S^+, S^-) \in Y$,

$$(8) \quad \|\mu_S\|_{TV} < \|\mu_{S^c}\|_{TV},$$

where $S = S^+ \cup S^-$.

— The **weak** nullspace property states as follows: For all nonzero measures μ in the nullspace of \mathcal{K}_n , and for all $(S^+, S^-) \in Y$,

$$\|\mu_S\|_{TV} \leq \|\mu_{S^c}\|_{TV},$$

where $S = S^+ \cup S^-$.

Given a nonzero measure μ in the nullspace of \mathcal{K}_n , this property means that more than half of the total variation of μ cannot be concentrated on a small subset. The nullspace property is a key to exact reconstruction as shown in the following proposition.

Proposition 4.1 — Let Y be a Jordan support family. Let σ be a signed measure having a Jordan support in Y . **If** the generalized moment morphism \mathcal{K}_n satisfies the nullspace property with respect to Y , **then**, the measure σ is the **unique** solution of generalized minimal extrapolation (GME) given the observation $\mathcal{K}_n(\sigma)$.

— **If** the generalized moment morphism \mathcal{K}_n satisfies the **weak** nullspace property with respect to Y , **then**, the measure σ is **a** solution of generalized minimal extrapolation (GME) given the observation $\mathcal{K}_n(\sigma)$.

Proof. See C.1. □

As far as we know, it is difficult to check the nullspace property. In the following, we give an example such that the weak nullspace property is satisfied.

4.3. The spaced out interpolation. We recall that S_Δ is the set of all pairs (S^+, S^-) of subsets of $I = [-1, 1]$ such that

$$(9) \quad \forall x, y \in S^+ \cup S^-, x \neq y, \quad |x - y| \geq \Delta.$$

The next lemma shows that if Δ is large enough then there exists a polynomial of degree n , with supremum norm not greater than 1, that interpolates 1 on the set S^+ and -1 on the set S^- .

Lemma 4.2 — *For all $(S^+, S^-) \in S_\Delta$, there exists a polynomial $P_{(S^+, S^-)}$ such that*

- $P_{(S^+, S^-)}$ has degree n not greater than $(2/\sqrt{\pi})(\sqrt{e}/\Delta)^{5/2+1/\Delta}$,
- $P_{(S^+, S^-)}$ is equal to 1 on the set S^+ ,
- $P_{(S^+, S^-)}$ is equal to -1 on the set S^- ,
- and $\|P_{(S^+, S^-)}\|_\infty \leq 1$ over I .

Proof. See C.2. □

This upper bound is meant to show that one can interpolate any sign sequence on S_Δ . Let us emphasize that this result is far from being sharp. Considering L_2 -minimizing polynomials under fitting constraint, the authors of the present paper believe that one can greatly improve the upper bound of Lemma 4.2. Indeed, our numerical experiments are in complete agreement with this comment. Invoking Lemma 1.1, Lemma 4.2 gives the next proposition.

Proposition 4.3 — *Let Δ be a positive real. If $n \geq (2/\sqrt{\pi})(\sqrt{e}/\Delta)^{5/2+1/\Delta}$ then \mathcal{K}_n satisfies the weak nullspace property with respect to S_Δ .*

Proof. See C.3. □

The bound $(2/\sqrt{\pi})(\sqrt{e}/\Delta)^{5/2+1/\Delta}$ can be considerably improved in actual practice. The following numerical experiment shows that this bound can be greatly lowered.

Some simulations. Our numerical experiment consists in looking for a generalized polynomial satisfying the assumption of Lemma 1.1. We work here with the cosine system $(1, \cos(\pi x), \cos(2\pi x), \dots, \cos(n\pi x))$ for various values of the integer n . As explained in Section 3, we can also consider the more classical power system $(1, x, x^2, \dots, x^n)$, so that our numerical experiments may be interpreted in this last frame. We consider signed measure having a support \mathcal{S} with $|\mathcal{S}| = 10$. We consider Δ -spaced out type measures for various values of Δ . For each choice of Δ , we draw uniformly 100 realizations of signed measures. This means that the points of \mathcal{S} are uniformly drawn on I^{10} , where $I = [0, 1)$ here, with the restriction that the minimal distance between two points is at least Δ and that there exist two points that are exactly Δ away from each other. Further, we uniformly randomized the signs of the measure on each point of \mathcal{S} . As we wish to work with *true* signed measures, we do not allow the case where all the signs are the same (negative or positive measures). Once we simulated the set S^+ and S^- , we wish to build an interpolating polynomial P of degree n having value 1 on S^+ , -1 on S^- and having a supremum norm minimum. As this last minimization is not obvious, we relax it to the minimization of the L_2 -norm with the extra restriction that the derivative of the interpolation polynomial vanishes on \mathcal{S} . Hence, when this

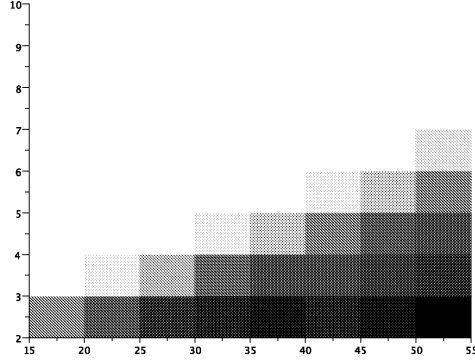


FIGURE 2. Consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ on $I = [0, 1]$. Set $s = 10$ the size of the target support. We are concerned with signed measures with Jordan support in S_Δ (see (9)). The abscissa represents the values of $1/\Delta$ (with $\Delta = 1/15, 1/20, \dots, 1/55$), and the ordinates represent the values of n (with $n = 20, 30, \dots, 100$). For each value of (Δ, n) , we draw uniformly 100 realizations of signed measures and the corresponding L_2 -minimizing polynomial P . The gray scale represents the percentage of times that $\|P\|_\infty \leq 1$ occurs. The white color means 100% ((GME) exactly recovers all the signed measures) while the black color represent 0% (in all our experiments, the polynomial P is such that $\|P\|_\infty > 1$ over I).

last optimization problem has a solution having a supremum norm not greater than 1, Lemma 1.1 may be applied and (GME) leads to exact reconstruction. The proportion of experimental results, where the supremum norm of the L_2 optimal polynomial is not greater than 1, is reported in Figure 2.

In our experiments we consider the values $\Delta = 1/15, 1/20, \dots, 1/55$. According to Proposition 4.3, the corresponding values of n range from 10^{19} to 10^{59} . In our experiments, we find that $n = 80$ suffices.

Acknowledgements: The authors would like to thank Jean Paul Calvi and Viet Hung Pham for fruitful comments. We also thank anonymous referees for their careful reviews and their interesting suggestions.

APPENDIX A. PROOFS OF SECTION 1.

A.1. Proof of Lemma 1.1. Assume that a generalized dual polynomial P exists. Let σ be such that $\sigma = \sum_{i=1}^s \sigma_i \delta_{x_i}$, with $\text{sgn}(\sigma_i) = \varepsilon_i$. Let σ^* be a solution of the generalized minimal extrapolation (GME) then $\int P d\sigma = \int P d\sigma^*$. The equality (ii) yields $\|\sigma\|_{TV} = \int P d\sigma$. Combining the two previous equalities,

$$\|\sigma\|_{TV} = \int P d\sigma = \int P d\sigma^* = \sum_{i=1}^s \varepsilon_i \sigma_i^* + \int P d\sigma_{S^c}^*,$$

where $\varepsilon_i = \text{sgn}(\sigma_i)$ and

$$\sigma^* = \sum_{i=1}^s \sigma_i^* \delta_{x_i} + \sigma_{\mathcal{S}^c}^*,$$

according to the Lebesgue decomposition (7). Since $\|P\|_\infty = 1$, we have

$$\sum_{i=1}^s \varepsilon_i \sigma_i^* + \int P d\sigma_{\mathcal{S}^c}^* \leq \|\sigma_{\mathcal{S}}^*\|_{TV} + \|\sigma_{\mathcal{S}^c}^*\|_{TV} = \|\sigma^*\|_{TV}.$$

Observe σ^* is a solution of (GME), it follows that $\|\sigma\|_{TV} = \|\sigma^*\|_{TV}$ and the above inequality is an equality. It yields $\int P d\sigma_{\mathcal{S}^c}^* = \|\sigma_{\mathcal{S}^c}^*\|_{TV}$. Moreover we have the following result.

Lemma A.1 — *Let $\nu \in \mathcal{M}$ with its support included in \mathcal{S}^c . If $\int P d\nu = \|\nu\|_{TV}$ then $\nu = 0$.*

Proof. Consider the compact set

$$\Omega_k = I \setminus \bigcup_{i=1}^s]x_i - \frac{1}{k}, x_i + \frac{1}{k}[, \quad \forall k > 0,$$

Suppose that there exists $k > 0$ such that $\|\nu_{\Omega_k}\|_{TV} \neq 0$. Then the inequality (iii) leads to $\int_{\Omega_k} P d\nu < \|\nu_{\Omega_k}\|_{TV}$. It yields

$$\|\nu\|_{TV} = \int P d\nu = \int_{\Omega_k} P d\nu + \int_{\Omega_k^c} P d\nu < \|\nu_{\Omega_k}\|_{TV} + \|\nu_{\Omega_k^c}\|_{TV} = \|\nu\|_{TV},$$

which is a contradiction. We deduce that $\|\nu_{\Omega_k}\|_{TV} = 0$, for all $k > 0$. The equality $\nu = 0$ follows with $\mathcal{S}^c = \bigcup_{k>0} \Omega_k$. \square

This lemma shows that σ^* is a discrete measure with its support included in \mathcal{S} . In this case, the moment constraint $\mathcal{K}_n(\sigma^* - \sigma) = 0$ can be written as a *generalized Vandermonde system*,

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & \dots & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix} \begin{pmatrix} \sigma_1^* - \sigma_1 \\ \sigma_2^* - \sigma_2 \\ \vdots \\ \sigma_s^* - \sigma_s \end{pmatrix} = 0.$$

From condition (i), we deduce that the generalized Vandermonde system is injective. \square

A.2. Proof of the remark in Section 1.2. Let σ belong to $\mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$. Consider the linear functional,

$$\Phi_f : \mu \mapsto \int_I f d\mu,$$

where f denotes a continuous bounded function. By definition, any subgradient Φ_f of the TV -norm at point σ satisfies, for all measures $\mu \in \mathcal{M}$,

$$\|\mu\|_{TV} - \|\sigma\|_{TV} \geq \Phi_f(\mu - \sigma).$$

Thus, one can easily check that f is equal to 1 (resp. -1) on $\text{supp}(\sigma^+)$ (resp. $\text{supp}(\sigma^-)$) and that $\|f\|_\infty = 1$. Conversely, any function f satisfying the latter condition leads to a subgradient Φ_f . Therefore, when it exists, the generalized

dual polynomial P is such that Φ_P is a *subgradient* of the TV-norm at point σ . Furthermore, let μ be a feasible point (i.e. $\mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)$). Since P is a generalized polynomial of order n , we deduce that $\Phi_P(\mu - \sigma) = 0$. Hence, the subgradient Φ_P is perpendicular to the set of feasible points. \square

APPENDIX B. PROOFS OF SECTION 2

B.1. Proof of Theorem 2.1. The proof essentially relies on Lemma 1.1. Let s be an integer. Let σ be a nonnegative measure. Let $\mathcal{S} = \{x_1, \dots, x_s\} \subset I$ be its support. The next lemma shows the existence of a generalized dual polynomial.

Lemma B.1 (Dual polynomial) — *Let s be an integer and n be such that $n = 2s$. Let \mathcal{F} be a homogeneous M -system on I . Let (x_1, \dots, x_s) be such that $\text{Index}(x_1, \dots, x_s) \leq n$. Then there exists a generalized polynomial P of degree d such that*

- (i) $s \leq d \leq n$,
- (ii) $P(x_i) = 1, \forall i = 1, \dots, s$,
- (iii) $|P(x)| < 1$ for all $x \notin \{x_1, \dots, x_s\}$.

We recall that Index is defined by (3). Note that these polynomials are presented in the first example of Definition 1.

Proof of Lemma B.1. Let (x_1, \dots, x_s) be such that $\text{Index}(x_1, \dots, x_s) \leq n$. From Lemma 2.2, there exists a nonnegative polynomial Q of degree d that vanishes exactly at the points x_i . Moreover, its degree d satisfies (i).

Since Q is continuous on the compact set \bar{I} , it is bounded and there exists a real c such that $\|Q\|_\infty < 1/c$. The generalized polynomial $P = 1 - cQ$ is the expected generalized polynomial. \square

Observe that

- Using Lemma B.1, it yields that there exists a generalized dual polynomial, of degree at most $n = 2s$, which interpolates the value 1 at points $\{x_1, \dots, x_s\}$.
- Since $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a T -system, the Vandermonde system given by (i) in Lemma 1.1 has full column rank.

Invoke Lemma 1.1 to conclude. \square

Remark. Since \mathcal{F} is a homogeneous M -system, the constant function 1 is a generalized polynomial. Note that the linear combination $P = 1 - cQ$ is a generalized polynomial because 1 is a generalized polynomial. This assumption is essential (see 2.2.2).

B.2. Proof of Proposition 2.3. Let $\sigma = \sum_{i=1}^s \sigma_i \delta_{x_i}$ be a nonnegative measure. Let $\mathcal{S} = \{x_1, \dots, x_s\}$ be its support. Let n be an integer such that $n \geq 2s$.

Step 1: Let $\mathcal{F}_h = \{1, u_1, u_2, \dots\}$ be a homogeneous M -system (the standard polynomials for instance). Let $t_1, \dots, t_{n+1} \in I \setminus \mathcal{S}$ be distinct points. It follows that the

Vandermonde system $\begin{pmatrix} 1 & \dots & 1 \\ u_1(t_1) & \dots & u_1(t_{n+1}) \\ \vdots & & \vdots \\ u_n(t_1) & \dots & u_n(t_{n+1}) \end{pmatrix}$ has full rank. Hence we may choose

$(v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+1}$ such that

- $v = \sum_{i=1}^{n+1} v_i \delta_{t_i}$,
- and for all $k = 0, \dots, n$, $\int_I u_k dv = \int_I u_k d\sigma$.

Step 2: Set

$$r = \frac{\|\sigma\|_{TV}}{\|v\|_{TV} + 1}.$$

Consider a positive continuous function u_0 such that

- $u_0(x_i) = r$, for $i = 1, \dots, s$,
- $u_0(t_i) = 1$, for $i = 1, \dots, n + 1$,
- the function u_0 is not constant.

Set $\mathcal{F} = \{u_0, u_0 u_1, u_0 u_2, \dots\}$. Obviously, \mathcal{F} is a non-homogeneous M -system. As usual, let \mathcal{K}_n denote the generalized moment morphism of order n derived from the family \mathcal{F} .

Last step: Set $\mu = r v$. An easy calculation gives $\mathcal{K}_n(\sigma) = \mathcal{K}_n(\mu)$. Note that

$$\|\mu\|_{TV} = \sum_{i=1}^{n+1} r |v_i| = \frac{\sum_{i=1}^{n+1} |v_i|}{\sum_{i=1}^{n+1} |v_i| + 1} \|\sigma\|_{TV} < \|\sigma\|_{TV}.$$

□

B.3. Proof of Theorem 2.4. Set $\mathcal{T} = \{t_1, \dots, t_p\}$. Let $\mathcal{M}_{\mathcal{T}}$ denote the set of all finite measures of which support is included in \mathcal{T} . Let $\Theta_{\mathcal{T}}$ be the linear map defined by

$$\Theta_{\mathcal{T}} : \begin{cases} (\mathbb{R}^p, \ell_1) & \rightarrow (\mathcal{M}_{\mathcal{T}}, \|\cdot\|_{TV}) \\ (x_1, \dots, x_p) & \mapsto \sum_{i=1}^p x_i \delta_{t_i} \end{cases}.$$

One can check that $\Theta_{\mathcal{T}}$ is a bijective isometry. Moreover, it holds that

$$(10) \quad \forall y \in \mathbb{R}^p, \quad \mathcal{K}_n(\Theta_{\mathcal{T}}(y)) = Ay,$$

where A is the generalized Vandermonde system defined by

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ u_1(t_1) & u_1(t_2) & \dots & u_1(t_p) \\ u_2(t_1) & u_2(t_2) & \dots & u_2(t_p) \\ \vdots & \vdots & & \vdots \\ u_n(t_1) & u_n(t_2) & \dots & u_n(t_p) \end{pmatrix}.$$

In the meantime, let x_0 be a nonnegative s -sparse vector. Let $\sigma = \Theta_{\mathcal{T}}(x_0)$. Observe that the support size of σ is at most s . Consequently, Theorem 2.1 shows that σ is the unique solution to (GME). Since $\sigma \in \mathcal{M}_{\mathcal{T}}$, we have that σ is the unique solution to the following program:

$$\sigma = \underset{\mu \in \mathcal{M}_{\mathcal{T}}}{\text{Arg min}} \|\mu\|_{TV} \quad \text{s.t. } \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma).$$

Using (10) and the isometry $\Theta_{\mathcal{T}}$, it follows that x_0 is the unique solution to the program:

$$x_0 = \underset{y \in \mathbb{R}^p}{\text{Arg min}} \|y\|_1 \quad \text{s.t. } Ay = Ax_0.$$

□

APPENDIX C. PROOFS OF SECTION 4

C.1. Proof of Proposition 4.1. Let \mathcal{K}_n be a generalized moment morphism that satisfies the nullspace property with respect to a Jordan support family \mathcal{Y} . Let σ be a signed measure of which Jordan support belongs to \mathcal{Y} . Let σ^* be a solution of (GME). Observe that $\|\sigma^*\|_{TV} \leq \|\sigma\|_{TV}$. Denote $\mu = \sigma^* - \sigma$ and note that $\mu \in \ker(\mathcal{K}_n)$. Then

$$\begin{aligned} \|\sigma^*\|_{TV} &= \|\sigma_S^*\|_{TV} + \|\sigma_{S^c}^*\|_{TV}, \\ &= \|\sigma + \mu_S\|_{TV} + \|\mu_{S^c}\|_{TV}, \\ &\geq \|\sigma\|_{TV} - \|\mu_S\|_{TV} + \|\mu_{S^c}\|_{TV}, \end{aligned}$$

where S denotes the support of σ . Suppose that $\mu \neq 0$. The nullspace property yields that the measure μ satisfies inequality (8). We deduce $\|\sigma^*\|_{TV} > \|\sigma\|_{TV}$, which is a contradiction. Thus $\mu = 0$ and $\sigma^* = \sigma$. □

C.2. Proof of Lemma 4.2. For sake of readability, we sketch the proof here. Let $(S^+, S^-) \in S_\Delta$. Set $S = S^+ \cup S^- = \{x_1, \dots, x_s\}$. Consider the Lagrange interpolation polynomials

$$l_k(x) = \frac{\prod_{i \neq k} (x - x_i)}{\prod_{i \neq k} (x_k - x_i)},$$

for $1 \leq k \leq s$. One can bound the supremum norm of l_k over $[0, 1]$ by

$$\|l_k\|_\infty \leq L(\Delta),$$

where $L(\Delta)$ is an upper bound that depends only on Δ . Consider the m -th Chebyshev polynomial of the first order $T_m(x) = \cos(m \arccos(x))$, for all $x \in [-1, 1]$. For a sufficiently large value of m , there exist $2s$ extrema ζ_i of T_m such that $|\zeta_i| \leq 1/(sL(\Delta))$. Interpolating values ζ_i at point x_k , we build the expected polynomial P . We find that the polynomial P has degree not greater than

$$C(\sqrt{e}/\Delta)^{5/2+1/\Delta},$$

where $C = 2/\sqrt{\pi}$. □

C.3. Proof of Proposition 4.3. Let μ be a nonzero measure in the nullspace of \mathcal{K}_n and $(\mathcal{A}, \mathcal{B})$ be in S_Δ . Let S be equal to $\mathcal{A} \cup \mathcal{B}$. Let S^+ (resp. S^-) be the set of points x in S such that the μ -weight at point x is nonnegative (resp. negative). Observe that $S = S^+ \cup S^-$ and $(S^+, S^-) \in S_\Delta$. From Lemma 4.2, there exists $P_{(S^+, S^-)}$ of degree not greater than n such that $P_{(S^+, S^-)}$ is equal to 1 on S^+ , -1 on S^- , and $\|P_{(S^+, S^-)}\|_\infty \leq 1$. It yields

$$\int P_{(S^+, S^-)} d\mu = \|\mu_S\|_{TV} + \int_{S^c} P_{(S^+, S^-)} d\mu \geq \|\mu_S\|_{TV} - \|\mu_{S^c}\|_{TV}.$$

Since $\mu \in \ker(\mathcal{K}_n)$, it follows that $\int P_{(S^+, S^-)} d\mu = 0$. □

APPENDIX D. NUMERICAL EXPERIMENTS

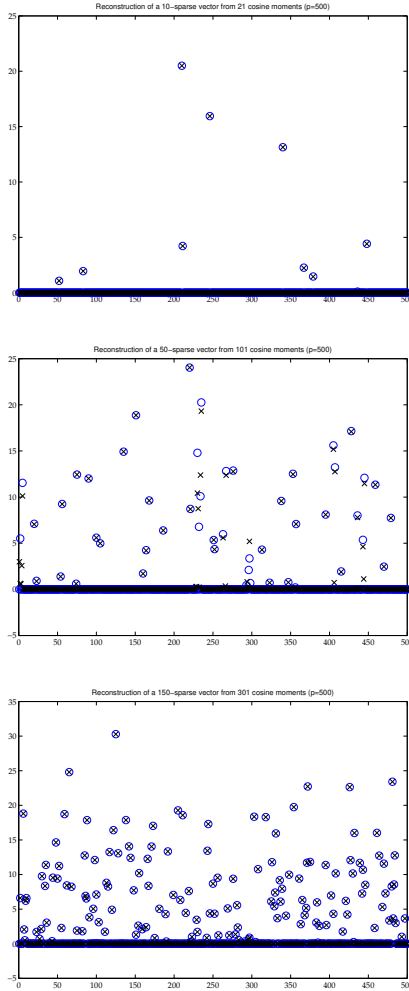


FIGURE 3. These numerical experiments illustrate Theorem 2.4. We consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ and the points $t_k = k/(p+1)$, for $k = 1, \dots, p$. The blue circles represent the target vector x_0 , while the black crosses represent the solution x^* of (BP). The respective values are $s = 10$, $n = 21$, $p = 500$; $s = 50$, $n = 101$, $p = 500$; and $s = 150$, $n = 301$, $p = 500$.

Note that some coefficients can be badly estimated (for instance when $s = 50$ and $n = 101$). This might be due to the fact that we consider the limit case $n = 2s + 1$. Nevertheless, this is not the case when we have very few coefficients ($s = 10$ and $n = 21$) or a large number of moments ($s = 150$ and $n = 301$). As a general rule, we observe faithful reconstruction.

REFERENCES

- [BE95] P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities*, Graduate Texts in Mathematics, vol. 161, Springer-Verlag, New York, 1995. MR 1367960 (97e:41001)
- [Beu38] A. Beurling, *Sur les intégrales de fourier absolument convergentes et leur application à une transformation fonctionnelle*, Ninth Scandinavian Mathematical Congress, 1938, pp. 345–366.
- [BEZ94] P. Borwein, T. Erdélyi, and J. Zhang, *Chebyshev polynomials and Markov-Bernstein type inequalities for rational spaces*, J. London Math. Soc. (2) **50** (1994), no. 3, 501–519. MR 1299454 (95j:41015)
- [BGI⁺08] R. Berinde, A. Gilbert, P. Indyk, H. Karloff, and M. Strauss, *Combining geometry and combinatorics: a unified approach to sparse signal recovery*.
- [Cal93] J. P. Calvi, *Polynomial interpolation with prescribed analytic functionals*, J. Approx. Theory **75** (1993), no. 2, 136–156. MR 1249394 (94j:41002)
- [Car07] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen*, Mathematische Annalen **64** (1907), no. 1, 95–115.
- [Car11] ———, *Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen*, Rendiconti del Circolo Matematico di Palermo (1884-1940) **32** (1911), no. 1, 193–217.
- [CDD09] A. Cohen, W. Dahmen, and R. DeVore, *Compressed sensing and best k -term approximation*, J. Amer. Math. Soc. **22** (2009), no. 1, 211–231. MR 2449058 (2010d:94024)
- [CDS98] S. S. Chen, D. L. Donoho, and M. A. Saunders, *Atomic decomposition by basis pursuit*, SIAM J. Sci. Comput. **20** (1998), no. 1, 33–61. MR 1639094 (99h:94013)
- [CDS01] ———, *Atomic decomposition by basis pursuit*, SIAM Rev. **43** (2001), no. 1, 129–159, Reprinted from SIAM J. Sci. Comput. **20** (1998), no. 1, 33–61 (electronic) [MR1639094 (99h:94013)]. MR 1854649
- [CP10] E. J. Candès and Y. Plan, *A probabilistic and ripples theory of compressed sensing*, arXiv (2010).
- [CRT06a] E. J. Candès, J. K. Romberg, and T. Tao, *Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information*, IEEE Trans. Inform. Theory **52** (2006), no. 2, 489–509. MR 2236170 (2007e:94020)
- [CRT06b] ———, *Stable signal recovery from incomplete and inaccurate measurements*, Comm. Pure Appl. Math. **59** (2006), no. 8, 1207–1223. MR 2230846 (2007f:94007)
- [Der56] D. Derry, *Convex hulls of simple space curves*, Canad. J. Math **8** (1956), 383–388.
- [DG96] P. Doukhan and F. Gamboa, *Superresolution rates in Prokhorov metric*, Canad. J. Math. **48** (1996), no. 2, 316–329. MR 1393035 (97j:44010)
- [DJHS92] D. L. Donoho, I. M. Johnstone, J. C. Hoch, and A. S. Stern, *Maximum entropy and the nearly black object*, Journal of the Royal Statistical Society. Series B (Methodological) (1992), 41–81.
- [Don92] D. L. Donoho, *Superresolution via sparsity constraints*, SIAM journal on mathematical analysis **23** (1992), 1309.
- [Don06] ———, *Compressed sensing*, IEEE Trans. Inform. Theory **52** (2006), no. 4, 1289–1306. MR 2241189 (2007e:94013)
- [DS89] D. L. Donoho and P. B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math. **49** (1989), no. 3, 906–931. MR 997928 (90c:42003)
- [DT05] D. L. Donoho and J. Tanner, *Sparse nonnegative solution of underdetermined linear equations by linear programming*, Proceedings of the National Academy of Sciences of the United States of America **102** (2005), no. 27, 9446.
- [DT09] ———, *Counting faces of randomly projected polytopes when the projection radically lowers dimension*, J. Amer. Math. Soc. **22** (2009), no. 1, 1–53. MR 2449053 (2009k:52015)
- [DT10] ———, *Counting the faces of randomly-projected hypercubes and orthants, with applications*, Discrete & Computational Geometry **43** (2010), no. 3, 522–541.
- [Fel68] W. Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons Inc., New York, 1968. MR 0228020 (37 #3604)
- [Fel71] ———, *An introduction to probability theory and its applications. Vol. II.*, Second edition, John Wiley & Sons Inc., New York, 1971. MR 0270403 (42 #5292)
- [Fuc96] J.-J. Fuchs, *Linear programming in spectral estimation. application to array processing*, Acoustics, Speech, and Signal Processing, 1996. ICASSP-96. Conference Proceedings., 1996 IEEE International Conference on, vol. 6, IEEE, 1996, pp. 3161–3164.

- [Fuc04] ———, *On sparse representations in arbitrary redundant bases*, IEEE Trans. Inform. Theory **50** (2004), no. 6, 1341–1344. MR 2094894 (2005e:94022)
- [Fuc05] ———, *Sparsity and uniqueness for some specific underdetermined systems*, IEEE International Conference on Acoustics, Speech, and Signal Processing, 2005.
- [GG96] F. Gamboa and E. Gassiat, *Sets of superresolution and the maximum entropy method on the mean*, SIAM J. Math. Anal. **27** (1996), no. 4, 1129–1152. MR 1393430 (97j:44009)
- [Hau21a] F. Hausdorff, *Summationsmethoden und Momentfolgen. I*, Math. Z. **9** (1921), no. 1-2, 74–109. MR 1544453
- [Hau21b] ———, *Summationsmethoden und Momentfolgen. II*, Math. Z. **9** (1921), no. 3-4, 280–299. MR 1544467
- [HSS96] Z. He, W. J. Studden, and D. Sun, *Optimal designs for rational models*, Ann. Statist. **24** (1996), no. 5, 2128–2147. MR 1421165 (98b:62137)
- [IS01] L. A. Imhof and W. J. Studden, *E-optimal designs for rational models*, Ann. Statist. **29** (2001), no. 3, 763–783. MR 1865340 (2002h:62222)
- [KN77] M. G. Kreĭn and A. A. Nudel'man, *The Markov moment problem and extremal problems*, American Mathematical Society, Providence, R.I., 1977, Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs, Vol. 50. MR MR0458081 (56 #16284)
- [KS53] S. Karlin and L. S. Shapley, *Geometry of moment spaces*, Mem. Amer. Math. Soc. **1953** (1953), no. 12, 93. MR 0059329 (15,512c)
- [KS66] S. Karlin and W. J. Studden, *Tchebycheff systems: With applications in analysis and statistics*, Pure and Applied Mathematics, Vol. XV, Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1966. MR 0204922 (34 #4757)
- [Rie11] F. Riesz, *Sur certains systèmes singuliers d'équations intégrales*, Ann. Sci. École Norm. Sup. (3) **28** (1911), 33–62. MR MR1509135
- [Riv90] T. J. Rivlin, *Chebyshev polynomials*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1990, From approximation theory to algebra and number theory. MR 1060735 (92a:41016)
- [Stu88] B. Sturmfels, *Totally positive matrices and cyclic polytopes*, Linear Algebra and its Applications **107** (1988), 275–281, Proceedings of the Victoria Conference on Combinatorial Matrix Analysis (Victoria, BC, 1987). MR 89i:52009

INSTITUT DE MATHÉMATIQUES DE TOULOUSE (CNRS UMR 5219). UNIVERSITÉ PAUL SABATIER, 118 ROUTE DE NARBONNE, 31062 TOULOUSE, FRANCE.

E-mail address: yohann.decastro@math.univ-toulouse.fr

E-mail address: fabrice.gamboa@math.univ-toulouse.fr