Asymptotic behavior of stochastic algorithms with statistical applications
Part 1

Bernard Bercu

University of Bordeaux, France

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Outline

1. Introduction
   - Sample mean and variance.
   - Recursive estimation of mean and variance.
   - Quantile of a continuous distribution.
   - Recursive estimation of quantile.

2. Convergence of martingales
   - Definition and Examples.
   - On Doob's convergence theorem.
   - Square integrable martingales.
   - Robbins-Siegmund theorem.
   - Strong law of large numbers for martingales.
   - Central limit theorem for martingales.

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Sample mean and variance.

Let (X_n, X) be a sequence of square integrable **independent and identically distributed** random variables with $\mathbb{E}[X] = m$, $\mathrm{Var}(X) = \sigma^2$. The **sample mean** and the **sample variance** are defined by

$$\overline{X}_n = \frac{1}{n} \sum_{k=1}^{n} X_k,$$

and

$$S_n^2 = \frac{1}{n} \sum_{k=1}^{n} (X_k - \overline{X}_n)^2.$$

Goal

$\longrightarrow$ Recursively estime the unknown mean and variance

$$\theta = \begin{pmatrix} m \\ \sigma^2 \end{pmatrix}.$$

Outline

Two recursive equations.

We have

$$(n+1)\overline{X}_{n+1} = \sum_{k=1}^{n} X_k + X_{n+1}.$$

Consequently,

$$(n+1)\overline{X}_{n+1} = n\overline{X}_n + X_{n+1} = (n+1)\overline{X}_n + X_{n+1} - \overline{X}_n,$$

which implies that

$$\boldsymbol{\overline{X}_{n+1} = \overline{X}_n + \frac{1}{n+1}\Big(X_{n+1} - \overline{X}_n\Big).}$$

We also have

$$S_n^2 = \frac{1}{n}\sum_{k=1}^{n} X_k^2 - \overline{X}_n^2.$$

Consequently,

$$\begin{aligned}(n+1)S_{n+1}^2 &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_{n+1}^2,\\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\Big(\overline{X}_n + \frac{1}{n+1}\big(X_{n+1} - \overline{X}_n\big)\Big)^2,\\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_n^2 - 2\overline{X}_n\big(X_{n+1} - \overline{X}_n\big) - \xi_{n+1}\end{aligned}$$

where

$$\xi_{n+1} = \frac{1}{n+1}\big(X_{n+1} - \overline{X}_n\big)^2.$$

We also have

$$S_n^2 = \frac{1}{n}\sum_{k=1}^{n} X_k^2 - \overline{X}_n^2.$$

Consequently,

$$\begin{aligned}(n+1)S_{n+1}^2 &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_{n+1}^2,\\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\Big(\overline{X}_n + \frac{1}{n+1}\Big(X_{n+1} - \overline{X}_n\Big)\Big)^2,\\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_n^2 - 2\overline{X}_n\Big(X_{n+1} - \overline{X}_n\Big) - \xi_{n+1}\end{aligned}$$

where

$$\xi_{n+1} = \frac{1}{n+1}\Big(X_{n+1} - \overline{X}_n\Big)^2.$$

We also have

$$S_n^2 = \frac{1}{n}\sum_{k=1}^{n} X_k^2 - \overline{X}_n^2.$$

Consequently,

$$\begin{aligned}(n+1)S_{n+1}^2 &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_{n+1}^2, \\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\Big(\overline{X}_n + \frac{1}{n+1}\Big(X_{n+1} - \overline{X}_n\Big)\Big)^2, \\ &= \sum_{k=1}^{n+1} X_k^2 - (n+1)\overline{X}_n^2 - 2\overline{X}_n\Big(X_{n+1} - \overline{X}_n\Big) - \xi_{n+1}\end{aligned}$$

where

$$\xi_{n+1} = \frac{1}{n+1}\Big(X_{n+1} - \overline{X}_n\Big)^2.$$

Two recursive equations.

Therefore,

$$(n+1)S_{n+1}^2 = \sum_{k=1}^{n} X_k^2 - n\overline{X}_n^2 + X_{n+1}^2 - 2\overline{X}_n X_{n+1} + \overline{X}_n^2 - \xi_{n+1},$$

$$= nS_n^2 + \left(X_{n+1} - \overline{X}_n\right)^2 - \xi_{n+1}.$$

Hence

$$(n+1)S_{n+1}^2 = (n+1)S_n^2 + \left(X_{n+1} - \overline{X}_n\right)^2 - S_n^2 - \xi_{n+1},$$

leading to

$$S_{n+1}^2 = S_n^2 + \frac{1}{n+1}\left(\left(X_{n+1} - \overline{X}_n\right)^2 - S_n^2\right) - \frac{1}{(n+1)^2}\left(X_{n+1} - \overline{X}_n\right)^2.$$

Two recursive equations.

Therefore,

$$
\begin{aligned}
(n+1)S_{n+1}^2 &= \sum_{k=1}^{n} X_k^2 - n\overline{X}_n^2 + X_{n+1}^2 - 2\overline{X}_n X_{n+1} + \overline{X}_n^2 - \xi_{n+1}, \\
&= nS_n^2 + \left(X_{n+1} - \overline{X}_n\right)^2 - \xi_{n+1}.
\end{aligned}
$$

Hence

$$
(n+1)S_{n+1}^2 = (n+1)S_n^2 + \left(X_{n+1} - \overline{X}_n\right)^2 - S_n^2 - \xi_{n+1},
$$

leading to

$$
\mathbf{S_{n+1}^2 = S_n^2 + \frac{1}{n+1}\left(\left(X_{n+1} - \overline{X}_n\right)^2 - S_n^2\right) - \frac{1}{(n+1)^2}\left(X_{n+1} - \overline{X}_n\right)^2.}
$$

A recursive matrix equation.

Denote

$$\widehat{\theta}_n = \begin{pmatrix} \overline{X}_n \\ S_n^2 \end{pmatrix}.$$

It follows from the previous calculation that

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \frac{1}{n+1} F(\widehat{\theta}_n, X_{n+1}) + \frac{1}{(n+1)} r_{n+1}$$

where

$$F(\widehat{\theta}_n, X_{n+1}) = \begin{pmatrix} X_{n+1} - \overline{X}_n \\ (X_{n+1} - \overline{X}_n)^2 - S_n^2 \end{pmatrix}$$

and

$$r_{n+1} = \begin{pmatrix} 0 \\ -\frac{1}{(n+1)} (X_{n+1} - \overline{X}_n)^2 \end{pmatrix}.$$

A recursive matrix equation.

However,

$$\mathbb{E}\big[F(\widehat{\theta}_n, X_{n+1})|\mathcal{F}_n\big] = f(\widehat{\theta}_n) + s_n$$

where $f(\widehat{\theta}_n) = \theta - \widehat{\theta}_n$ and

$$s_n = \begin{pmatrix} 0 \\ (m - \overline{X}_n)^2 \end{pmatrix}.$$

Consequently, we obtain the **martingale decomposition**

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n + \frac{1}{n+1}\Big(f(\widehat{\theta}_n) + \varepsilon_{n+1} + R_{n+1}\Big)$$

where (ε_n) is a **martingale difference sequence,** $\mathbb{E}[\varepsilon_{n+1}|\mathcal{F}_n] = 0$ and the remainder $R_{n+1} = r_{n+1} + s_n$ is negligeable.

A first warm-up result.

Theorem

Assume that (X_n, X) *is a sequence of* **iid** *random variables such that* $\mathbb{E}[X^4]$ *is finite. Denote* $\mathbb{E}[(X-m)^3] = \mu^3$ *and* $\mathbb{E}[(X-m)^4] = \tau^4$. *Then, we have the almost sure convergence*

$$\lim_{n\to\infty} \widehat{\theta}_n = \theta \qquad \text{a.s.}$$

In addition, we also have the asymptotic normality

$$\sqrt{n}\Big(\widehat{\theta}_n - \theta\Big) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Gamma)$$

where

$$\Gamma = \begin{pmatrix} \sigma^2 & \mu^3 \\ \mu^3 & \tau^4 - \sigma^4 \end{pmatrix}.$$

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Quantile of a continuous distribution.

Let X be a **continuous** random variable with **unknown** distribution function F. Assume that F is **continuous and strictly increasing.**

Definition

For any α in $]0,1[$, the quantile of order α of X is the unique solution θ_α of the equation $F(x) = \alpha$,

$$\boldsymbol{F(\theta_\alpha) = \alpha.}$$

For the Exponential $\mathcal{E}(\lambda)$ distribution with $\lambda > 0$,

$$\theta_\alpha = -\frac{1}{\lambda}\log(1-\alpha).$$

Goal

$\longrightarrow$ Recursively estime the unknown quantile $\boldsymbol{\theta_\alpha}$.

Outline

Let (X_n) be a sequence of **iid** random variables sharing the same distribution as X. We estimate θ_α by the recursive estimator

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n - \frac{1}{n+1}\Big(Y_{n+1} - \alpha\Big)$$

where

$$Y_{n+1} = F\big(\widehat{\theta}_n, X_{n+1}\big) = \mathrm{I}_{\{X_{n+1} \leqslant \widehat{\theta}_n\}}.$$

We clearly have $\mathbb{E}\big[Y_{n+1}|\mathcal{F}_n\big] = F\big(\widehat{\theta}_n\big)$ leading to the **martingale decomposition**

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n - \frac{1}{n+1}\Big(F\big(\widehat{\theta}_n\big) + \varepsilon_{n+1} - \alpha\Big)$$

where (ε_n) is a **martingale difference sequence,** $\mathbb{E}[\varepsilon_{n+1}|\mathcal{F}_n] = 0$.

A second warm-up result.

Denote by f the probability density function of X.

Theorem

We have the almost sure convergence

$$\lim_{n\to\infty} \widehat{\theta}_n = \theta_\alpha \qquad \text{a.s.}$$

Moreover, as soon as $\mathbf{f(\theta_\alpha) > 1/2}$*, we have the asymptotic normality*

$$\sqrt{n}\Big(\widehat{\theta}_n - \theta_\alpha\Big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{\alpha(1-\alpha)}{2f(\theta_\alpha)-1}\Big).$$

A second warm-up result.

Consider the **slow down Robbins-Monro algorithm** given by

$$\widehat{\theta}_{n+1} = \widehat{\theta}_n - \gamma_n \Big(Y_{n+1} - \alpha \Big)$$

where

$$\gamma_n = \frac{1}{n^c} \qquad \text{with} \qquad \frac{1}{2} < c < 1.$$

At time $n \geqslant 1$, compute de Cesaro mean

$$\overline{\theta}_n = \frac{1}{n} \sum_{k=1}^{n} \widehat{\theta}_k.$$

A second warm-up result.

We already saw that

$$\overline{\theta}_{n+1} = \overline{\theta}_n + \frac{1}{n+1}\Big(\widehat{\theta}_{n+1} - \overline{\theta}_n\Big).$$

Theorem

We have the almost sure convergence

$$\lim_{n\to\infty} \overline{\theta}_n = \theta_\alpha \qquad \text{a.s.}$$

Moreover, we also have the asymptotic normality

$$\sqrt{n}\Big(\overline{\theta}_n - \theta_\alpha\Big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(0, \frac{\alpha(1-\alpha)}{f^2(\theta_\alpha)}\Big).$$

Outline

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_n)$ where $\mathcal{F}_n$ is the σ-algebra of events occurring up to time n.

Definition

Let (M_n) be a sequence of integrable random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that, for all $n \geqslant 0$, M_n is $\mathcal{F}_n$-measurable.

1. (M_n) is a martingale **MG** if for all $n \geqslant 0$,
$$\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = M_n \qquad \text{a.s.}$$
2. (M_n) is a submartingale **sMG** if for all $n \geqslant 0$,
$$\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \geqslant M_n \qquad \text{a.s.}$$
3. (M_n) is a supermartingale **SMG** if for all $n \geqslant 0$,
$$\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] \leqslant M_n \qquad \text{a.s.}$$

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Martingales with sums.

Example (Sums)

Let (X_n) be a sequence of integrable and independent random variables such that, for all $n \geqslant 1$, $\mathbb{E}[X_n] = m$. Denote

$$S_n = \sum_{k=1}^{n} X_k.$$

We clearly have

$$S_{n+1} = S_n + X_{n+1}.$$

Consequently, (S_n) is a sequence of integrable random variables with

$$\begin{aligned} \mathbb{E}[S_{n+1} \mid \mathcal{F}_n] &= S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n], \\ &= S_n + \mathbb{E}[X_{n+1}], \end{aligned}$$

Martingales with sums.

Example (Sums)

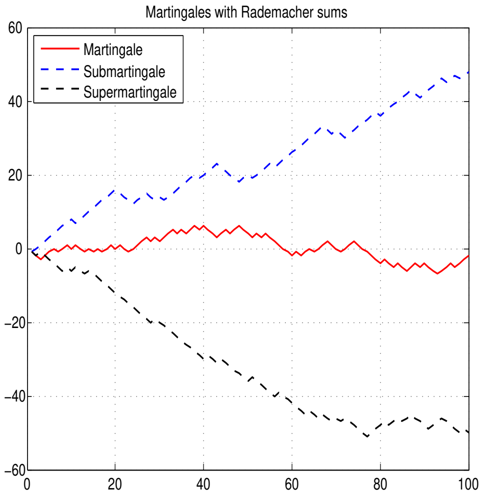
$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + m.$$

- (S_n) is a **martingale** if $m = 0$,
- (S_n) is a **submartingale** if $m \geqslant 0$,
- (S_n) is a **supermartingale** if $m \leqslant 0$.

$\longrightarrow$ It holds for Rademacher $\mathcal{R}(p)$ distribution with $0 < p < 1$ where

$$m = 2p - 1.$$

Martingales with Rademacher sums.



Martingales with products.

Example (Products)

Let (X_n) be a sequence of positive, integrable and independent random variables such that, for all $n \geqslant 1$, $\mathbb{E}[X_n] = m$. Denote

$$P_n = \prod_{k=1}^{n} X_k.$$

We clearly have

$$P_{n+1} = P_n X_{n+1}.$$

Consequently, (P_n) is a sequence of integrable random variables with

$$\begin{aligned} \mathbb{E}[P_{n+1} \mid \mathcal{F}_n] &= P_n \mathbb{E}[X_{n+1} \mid \mathcal{F}_n], \\ &= P_n \mathbb{E}[X_{n+1}], \end{aligned}$$

Martingales with products.

Example (Products)

$$\mathbb{E}[P_{n+1} \mid \mathcal{F}_n] = mP_n.$$

- (P_n) is a **martingale** if $m = 1$,
- (P_n) is a **submartingale** if $m \geqslant 1$,
- (P_n) is a **supermartingale** if $m \leqslant 1$.

$\longrightarrow$ It holds for Exponential $\mathcal{E}(\lambda)$ distribution with $\lambda > 0$ where

$$m = \frac{1}{\lambda}.$$

Outline

Doob's convergence theorem.

Theorem (Doob)

Let (M_n) *be a* **MG**, **sMG**, *or* **SMG** *bounded in* $\mathbb{L}^1$ *which means*

$$\sup_{n\geqslant 0} \mathbb{E}[|M_n|] < +\infty.$$

Then, we have the almost sure convergence

$$\lim_{n\to\infty} M_n = M_\infty \qquad a.s.$$

where M_∞ *is an integrable random variable.*

Joseph Leo Doob

Jacques Neveu

Convergence of martingales.

Theorem

Let (M_n) *be a* **MG** *bounded in* $\mathbb{L}^p$ *with* $p \geqslant 1$, *which means that*

$$\sup_{n \geqslant 0} \mathbb{E}[|M_n|^p] < +\infty.$$

1. *If* $p > 1$, (M_n) *converges almost surely to an integrable random variable* M_∞. *The convergence is also true in* $\mathbb{L}^p$.
2. *If* $p = 1$, (M_n) *converges almost surely to an integrable random variable* M_∞. *The convergence holds in* $\mathbb{L}^1$ **as soon as** (M_n) *is* **uniformly integrable** *that is*

$$\lim_{a \to \infty} \sup_{n \geqslant 0} \mathbb{E}\left[|M_n| \mathrm{I}_{\{|M_n| \geqslant a\}}\right] = 0.$$

Chow's Theorem.

Theorem (Chow)

Let (M_n) *be a* **MG** *such that for* $1 \leqslant a \leqslant 2$ *and for all* $n \geqslant 1$,

$$\mathbb{E}[|M_n|^a] < \infty.$$

Denote, for all $n \geqslant 1$, $\Delta M_n = M_n - M_{n-1}$ *and assume that*

$$\sum_{n=1}^{\infty} \mathbb{E}[|\Delta M_n|^a | \mathcal{F}_{n-1}] < \infty \qquad a.s.$$

Then, we have the almost sure convergence

$$\lim_{n \to \infty} M_n = M_\infty \qquad a.s.$$

where M_∞ *is an integrable random variable.*

Exponential Martingale.

Example (Exponential Martingale)

Let (X_n) be a sequence of independent random variable sharing the same $\mathcal{N}(0,1)$ distribution. For all $t \in \mathbb{R}^*$, let $S_n = X_1 + \cdots + X_n$ and denote

$$M_n(t) = \exp\Big(tS_n - \frac{nt^2}{2}\Big).$$

It is clear that $(M_n(t))$ is a **MG** which converges a.s. to zero. However, $\mathbb{E}[M_n(t)] = \mathbb{E}[M_1(t)] = 1$. It means that $(M_n(t))$ does not converge in $\mathbb{L}^1$.

Autoregressive Martingale.

Example (Autoregressive Martingale)

Let (X_n) be the autoregressive process given for all $n \geqslant 0$ by

$$X_{n+1} = \theta X_n + (1 - \theta)\varepsilon_{n+1}$$

where the initial state $X_0 = p$, $0 < p < 1$ and the parameter $0 < \theta < 1$. Assume that $\mathcal{L}(\varepsilon_{n+1} | \mathcal{F}_n)$ is the Bernoulli $\mathcal{B}(X_n)$ distribution. We can show that $0 < X_n < 1$ and that (X_n) is a **MG** satisfying

$$\lim_{n \to \infty} X_n = X_\infty \qquad \text{a.s.}$$

The convergence also holds in $\mathbb{L}^r$ for all $r \geqslant 1$. Finally, we can prove that X_∞ has the Bernoulli $\mathcal{B}(p)$ distribution.

Outline

Increasing process.

Definition

Let (M_n) be a square integrable **MG** that is for all $n \geqslant 1$,

$$\mathbb{E}[M_n^2] < \infty.$$

The **increasing process** associated with (M_n) is given by $< M >_0 = 0$ and, for all $n \geqslant 1$,

$$< M >_n = \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2 | \mathcal{F}_{k-1}]$$

where $\Delta M_k = M_k - M_{k-1}$.

Example (Increasing Process)

Let (X_n) be a sequence of square integrable and independent random variables such that, for all $n \geqslant 1$, $\mathbb{E}[X_n] = m$ and $\text{Var}(X_n) = \sigma^2 > 0$. Denote

$$M_n = \sum_{k=1}^{n} (X_k - m).$$

Then, (M_n) is a martingale and its increasing process reduces to

$$< M >_n = \sigma^2 n.$$

Outline

Theorem (Robbins-Siegmund)

Let (V_n), (A_n) and (B_n) be three positive sequences adapted to $\mathbb{F} = (\mathcal{F}_n)$. Assume that V_0 is integrable and, for all $n \geqslant 0$,

$$\mathbb{E}[V_{n+1}|\mathcal{F}_n] \leqslant V_n + A_n - B_n \qquad a.s.$$

Assume also that

$$\sum_{n=0}^{\infty} A_n < +\infty \qquad a.s.$$

1. *The sequence (V_n) converges a.s. to a random variable V_∞.*
2. *We also have*

$$\sum_{n=0}^{\infty} B_n < +\infty \qquad a.s.$$

Corollary

Let (V_n), (A_n), (B_n) and (a_n) be four positive sequences adapted to $\mathbb{F} = (\mathcal{F}_n)$. Assume that V_0 is integrable and, for all $n \geqslant 0$,

$$\mathbb{E}[V_{n+1}|\mathcal{F}_n] \leqslant V_n(1 + a_n) + A_n - B_n \qquad \text{a.s.}$$

Assume also that

$$\sum_{n=0}^{\infty} a_n < +\infty, \qquad \sum_{n=0}^{\infty} A_n < +\infty \qquad \text{a.s.}$$

1. *The sequence (V_n) converges a.s. to a random variable V_∞.*
2. *We also have*

$$\sum_{k=0}^{n} B_k < +\infty \qquad \text{a.s.}$$

Outline

Strong law of large numbers for martingales.

Theorem (Strong Law of large numbers)

Let (M_n) *be a square integrable* **MG** *and denote*

$$< M >_\infty = \lim_{n\to\infty} < M >_n .$$

1. *Assume that* $< M >_\infty < \infty$ *a.s. Then, we have*

$$\lim_{n\to\infty} M_n = M_\infty \qquad a.s.$$

2. *Assume that* $< M >_\infty = \infty$ *a.s. Then, we have*

$$\lim_{n\to\infty} \frac{M_n}{< M >_n} = 0 \qquad a.s.$$

$\longrightarrow$ If it exists a positive sequence (a_n) increasing to infinity such that $< M >_n = 0(a_n)$ a.s., then we have $M_n = o(a_n)$ a.s.

Strong law of large numbers for martingales, continued

Theorem (Strong Law of large numbers)

Let (M_n) be a square integrable **MG** *such that*

$$\lim_{n\to\infty} < M >_n = \infty \qquad a.s.$$

1. *For any positive γ, we have*

$$\frac{M_n^2}{< M >_n} = o\Big(\big(\log < M >_n\big)^{1+\gamma}\Big) \qquad a.s.$$

2. *If the increments of (M_n) have conditional moments of order* > 2,

$$\frac{M_n^2}{< M >_n} = O\big(\log < M >_n\big) \qquad a.s.$$

Example on sums.

Let (X_n) be a sequence of square integrable and independent random variables such that, for all $n \geqslant 1$, $\mathbb{E}[X_n] = m$ and $\text{Var}(X_n) = \sigma^2 > 0$. We already saw that

$$M_n = \sum_{k=1}^{n} (X_k - m)$$

is square integrable **MG** with $< M >_n = \sigma^2 n$. It follows from the **SLLN** for martingales that $M_n = o(n)$ a.s. which means that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k = m \qquad \text{a.s.}$$

More precisely, for any positive γ,

$$\Big(\frac{M_n}{n}\Big)^2 = \Big(\frac{1}{n} \sum_{k=1}^{n} X_k - m\Big)^2 = o\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \qquad \text{a.s.}$$

Outline

Central limit theorem for martingales.

Theorem (Central Limit Theorem)

Let (M_n) be a square integrable **MG** *and let (a_n) be a sequence of positive real numbers increasing to infinity. Assume that*

1. *It exists a deterministic limit $L \geqslant 0$ such that*

$$\frac{<M>_n}{a_n} \xrightarrow{\mathcal{P}} L.$$

2. **Lindeberg's condition.** *For all positive ε,*

$$\frac{1}{a_n}\sum_{k=1}^{n} \mathbb{E}[|\Delta M_k|^2 \mathrm{I}_{\{|\Delta M_k| \geqslant \varepsilon\sqrt{a_n}\}} | \mathcal{F}_{k-1}] \xrightarrow{\mathcal{P}} 0$$

where $\Delta M_k = M_k - M_{k-1}$.

Central limit theorem for martingales.

Theorem (Central Limit Theorem)

Let (M_n) be a square integrable **MG** *and let (a_n) be a sequence of positive real numbers increasing to infinity. Assume that*

1. *It exists a deterministic limit $L \geqslant 0$ such that*

$$\frac{<M>_n}{a_n} \xrightarrow{\mathcal{P}} L.$$

2. **Lindeberg's condition.** *For all positive ε,*

$$\frac{1}{a_n} \sum_{k=1}^{n} \mathbb{E}[|\Delta M_k|^2 \mathrm{I}_{\{|\Delta M_k| \geqslant \varepsilon \sqrt{a_n}\}} | \mathcal{F}_{k-1}] \xrightarrow{\mathcal{P}} 0$$

where $\Delta M_k = M_k - M_{k-1}$.

Central limit theorem fro martingales, continued.

Theorem (Central Limit Theorem)

Then, we have the asymptotic normality

$$\frac{1}{\sqrt{a_n}} M_n \xrightarrow{\mathcal{L}} \mathcal{N}(0, L).$$

Moreover, if $L > 0$, we also have

$$\sqrt{a_n}\Big(\frac{M_n}{<M>_n}\Big) \xrightarrow{\mathcal{L}} \mathcal{N}(0, L^{-1}).$$

$\longrightarrow$ Lyapunov's condition implies Lindeberg's condition : For $b > 2$,

$$\sum_{k=1}^{n} \mathbb{E}[|\Delta M_k|^b | \mathcal{F}_{k-1}] = O(a_n) \qquad \text{a.s.}$$