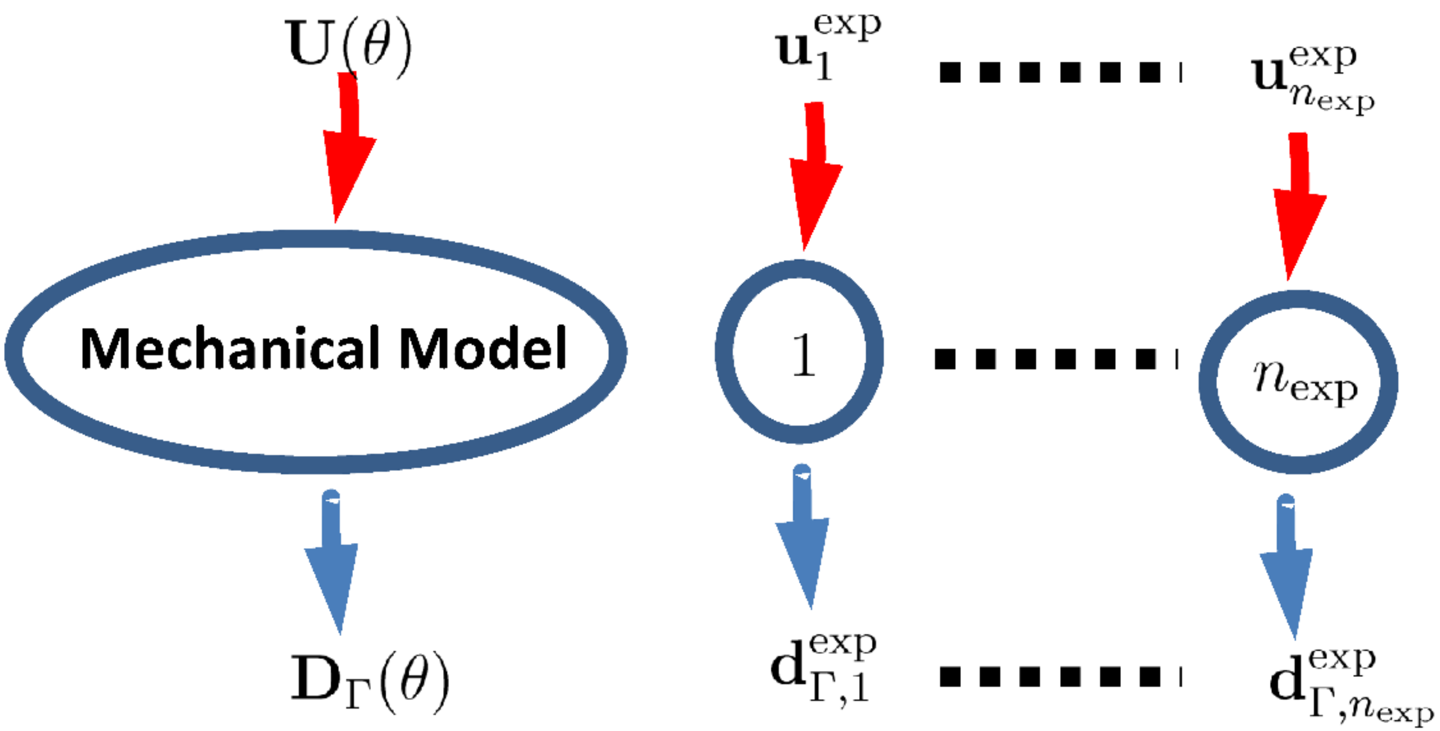
Maximum Likelihood Estimation of Stochastic Chaos Representations from Experimental Data

Christophe Desceliers, Christian Soize

Université Paris-Est, Laboratoire Modélisation et Simulation Multi Echelle, MSME
UMR 8208 CNRS, 5 Bd Descartes, 77454 Marne-la-Vallée, France

Laboratoire Modélisation
et Simulation Multi Echelle

Motivations



1. Construction of a probabilistic model for $\mathbf{U}$
2. No information on $\mathbf{U}$?
3. Experimental observations on the output are available

Outlines

Adapted functional basis

1. Second-order centered random field $\mathbf{U}(\mathbf{x}) = (U_1(\mathbf{x}), \ldots, U_n(\mathbf{x}))$
2. Let H be a Hilbert space of functions
3. Let $\{\mathbf{v}_j\}$ be an orthonormal hilbertian basis in H

Mean-squared representation

$$\mathbf{U}(\mathbf{x}) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} Q_j \mathbf{v}_j(\mathbf{x})$$

4. $Q_j = \frac{1}{\sqrt{\lambda_j}} \langle \mathbf{U}, \mathbf{v}_j \rangle_H$ are centrered and correlated

Karhunen-Loeve expansion

1. Hilbert space $H = L^2(\mathcal{B}, \mathbb{R}^n)$
2. Let $[R_{\mathbf{U}}(\mathbf{x}, \mathbf{x}')]$be the matrix of covariances of $\mathbf{U}(\mathbf{x})$
3. Eigenvalue problem $\mathbf{R_U v} = \lambda \mathbf{v}$
4. $\{Q_j\}$ are second-order centered orthogonal random variables
5. $\{Q_j\}$ are statistically dependent

Finite elements basis

Finite elements approximation

$$\mathbf{U}(\mathbf{x}) \simeq \sum_{k=1}^{N_i} \mathbf{U}_k \, h_k(\mathbf{x}) \simeq [H(\mathbf{x})] \, \mathbb{U}$$

1. $[H(\mathbf{x})]$ defines a metric matrix $[\mathbb{H}]$ such that

$$\langle \mathbf{v}, \mathbf{w} \rangle_H = \langle [\mathbb{H}] \, \mathbb{v}, \, \mathbb{w} \rangle_{\mathbb{R}^{n \times N_i}}$$

2. we have $Q_j = \dfrac{1}{\sqrt{\lambda_j}} \langle [\mathbb{H}] \, \mathbb{U} \, , \mathbb{v}_j \rangle_{\mathbb{R}^{n \times N_i}}$

FE approximation of the KL expansion

1. Let $[R_{\mathbb{U}}]$ be the covariance matrix of $\mathbb{U}$

   - $[R_{\mathbf{U}}(\mathbf{x},\mathbf{x}')] \simeq [H(\mathbf{x})]\,[R_{\mathbb{U}}]\,[H(\mathbf{x}')]^T$
   - $[R_{\mathbb{U}}] \simeq \int_{\mathcal{B}}\int_{\mathcal{B}} [\mathbb{H}]^{-1}\,[H(\mathbf{x})]^T\,[R_{\mathbf{U}}(\mathbf{x},\mathbf{x}')]\,[H(\mathbf{x}')]\,[\mathbb{H}]^{-1}\,d\mathbf{x}\,d\mathbf{x}'$

FE approximation of the eigenvalue problem

$$[\mathbb{H}]\,[R_{\mathbb{U}}]\,[\mathbb{H}]\,\mathbb{v}_j = \lambda_j [\mathbb{H}]\,\mathbb{v}_j$$

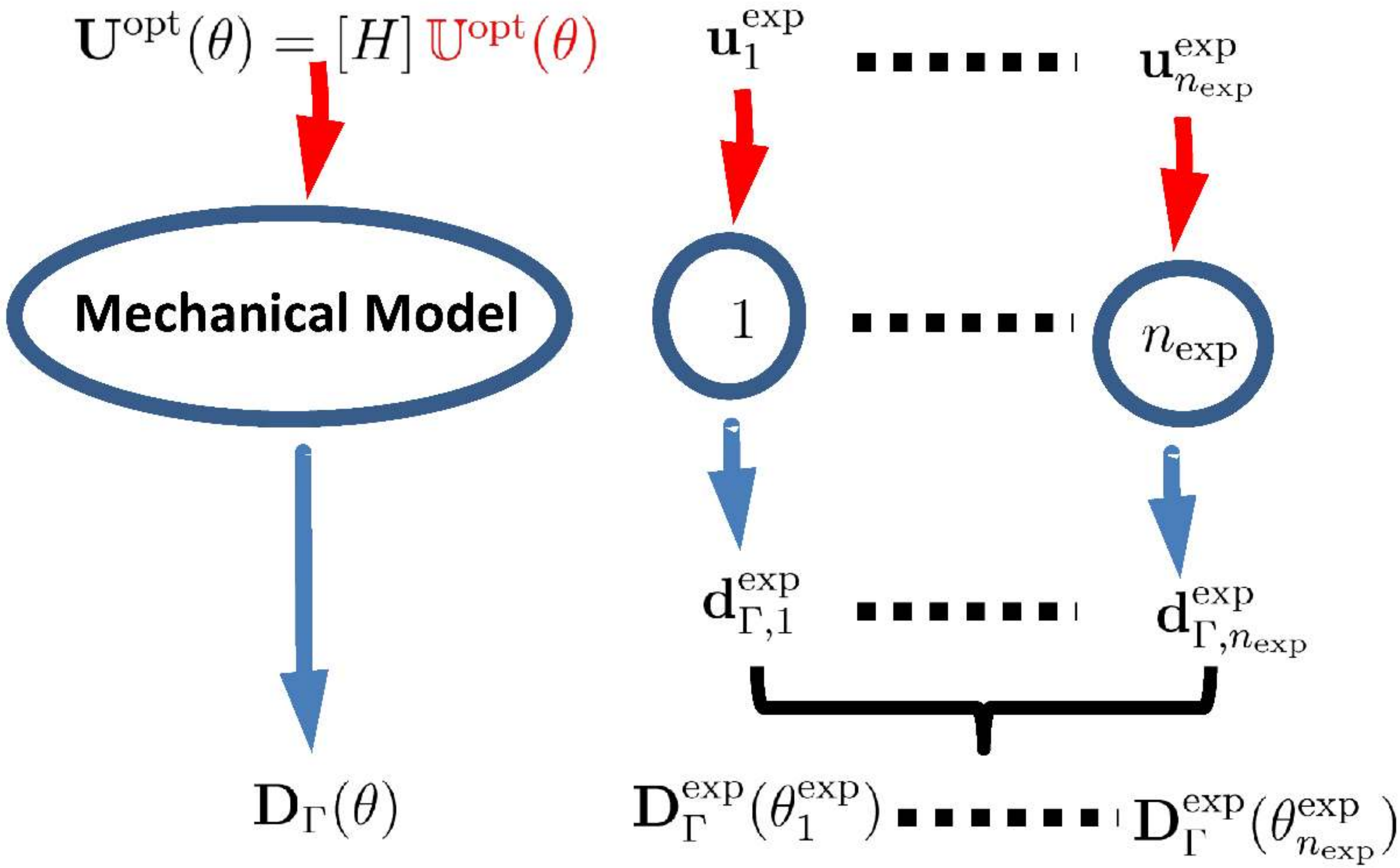
2. $Q_j = \dfrac{1}{\sqrt{\lambda_j}}\langle[\mathbb{H}]\,\mathbb{U}\,,\mathbb{v}_j\rangle_{\mathbb{R}^{n\times N_i}}$ are orthonormal set of second-order random variables

Limited information on the random field

knowledge on $[R_{\mathbf{U}}(\mathbf{x},\mathbf{x}')]$ or $[R_{\mathbb{U}}]$?

Outlines

KL expansion in solving an inverse problem



Modèle probabiliste de $\mathbb{U}^{\text{opt}}$

$$\mathbb{U}^{\text{opt}}(\theta) = \arg \min_{\mathbb{u}\in\mathbb{R}^{n\times N_i}} \left\{ \ell_{\det}(\mathbf{D}_\Gamma(\Theta); \mathbf{D}_\Gamma^{\exp}(\theta)) \right\}$$

KL expansion in solving an inverse problem

1.
$$\mathbb{U}^{\text{opt}}(\theta_1^{\text{exp}}) = \arg \min_{\mathbb{u} \in \mathbb{R}^{n \times N_i}} \left\{ \ell_{\text{det}}(\mathbf{D}_\Gamma(\theta_1^{\text{exp}}); \mathbf{D}_\Gamma^{\text{exp}}(\theta_1^{\text{exp}})) \right\}$$

⋮

$$\mathbb{U}^{\text{opt}}(\theta_{n_{\text{exp}}}^{\text{exp}}) = \arg \min_{\mathbb{u} \in \mathbb{R}^{n \times N_i}} \left\{ \ell_{\text{det}}(\mathbf{D}_\Gamma(\theta_{n_{\text{exp}}}^{\text{exp}}); \mathbf{D}_\Gamma^{\text{exp}}(\theta_{n_{\text{exp}}}^{\text{exp}})) \right\}$$

2.
$$[R_{\mathbb{U}}] = \frac{1}{n_{\text{exp}}} \sum_{j=1}^{n_{\text{exp}}} \mathbb{U}^{\text{opt}}(\theta_j^{\text{exp}}) \, \mathbb{U}^{\text{opt}}(\theta_j^{\text{exp}})^T$$

3.
$$[\mathbb{H}] \, [R_{\mathbb{U}}] \, [\mathbb{H}] \, \mathbb{v}_j = \lambda_j [\mathbb{H}] \, \mathbb{v}_j$$

4.
$$Q_j(\theta_1^{\text{exp}}) = \frac{1}{\sqrt{\lambda_j}} \langle [\mathbb{H}] \, \mathbb{U}^{\text{opt}}(\theta_1^{\text{exp}}) \, , \mathbb{v}_j \rangle_{\mathbb{R}^{n \times N_i}}$$

⋮

$$Q_j(\theta_{n_{\text{exp}}}^{\text{exp}}) = \frac{1}{\sqrt{\lambda_j}} \langle [\mathbb{H}] \, \mathbb{U}^{\text{opt}}(\theta_{n_{\text{exp}}}^{\text{exp}}) \, , \mathbb{v}_j \rangle_{\mathbb{R}^{n \times N_i}}$$

Outlines

1. Introduction
2. Construction of an adapted functional bases
3. Statistical reduction in solving an inverse problem
4. **Polynomial chaos expansion**
5. Stochastic solver
6. Example
7. Conclusions

Truncated polynomial chaos expansion

PCE of the random vector of generalized coordinates

- $$\mathbf{Q}^{\nu,d} = \sum_{\boldsymbol{\alpha},|\boldsymbol{\alpha}|=1}^{d} \mathbf{a}_{\boldsymbol{\alpha}} H_{\boldsymbol{\alpha}}(\mathbf{W}^{\nu})$$
- $$\mathbf{a}_{\boldsymbol{\alpha}} = E\{\mathbf{Q}\, H_{\boldsymbol{\alpha}}(\mathbf{W}^{\nu})\}$$

1. convergent for $\{W_i\}$ being an orthonormal set of centered Gaussian random variables.
2. direct estimate of $\mathbf{a}_{\boldsymbol{\alpha}}$ is not possible

Relation between the coefficients of the PCE

$$\sum_{\boldsymbol{\alpha},|\boldsymbol{\alpha}|=1}^{\infty} \mathbf{a}_{\boldsymbol{\alpha}}\, \mathbf{a}_{\boldsymbol{\alpha}}^{T} = [I_{\mu}]$$

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PCE of the random vector of generalized coordinates

$$\mathbb{a}^{\mathrm{opt}} = \arg \min_{\mathbb{a} \in \mathbb{R}^{\mu n_{\mathrm{chaos}}}} \left\{ L(\mathbb{a}) \quad \text{with} \quad \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}|=1}^{d} \mathbf{a}_{\boldsymbol{\alpha}} \mathbf{a}_{\boldsymbol{\alpha}}^T = [I_\mu] \right\}$$

- $\mathbb{a} = (\mathbf{a}_{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| = 1, \ldots, d)$

- $L(\mathbb{a}) = -\sum_{\ell=1}^{n_{\exp}} \log \left(p_{\mathbf{Q}^{\nu,d}}(\mathbf{Q}(\theta_\ell^{\exp}); \mathbb{a}) \right)$

- $\mathbf{Q}^{\nu,d} = \sum_{\boldsymbol{\alpha}, |\boldsymbol{\alpha}|=1}^{d} \mathbf{a}_{\boldsymbol{\alpha}} \, H_{\boldsymbol{\alpha}}(\mathbf{W}^\nu)$

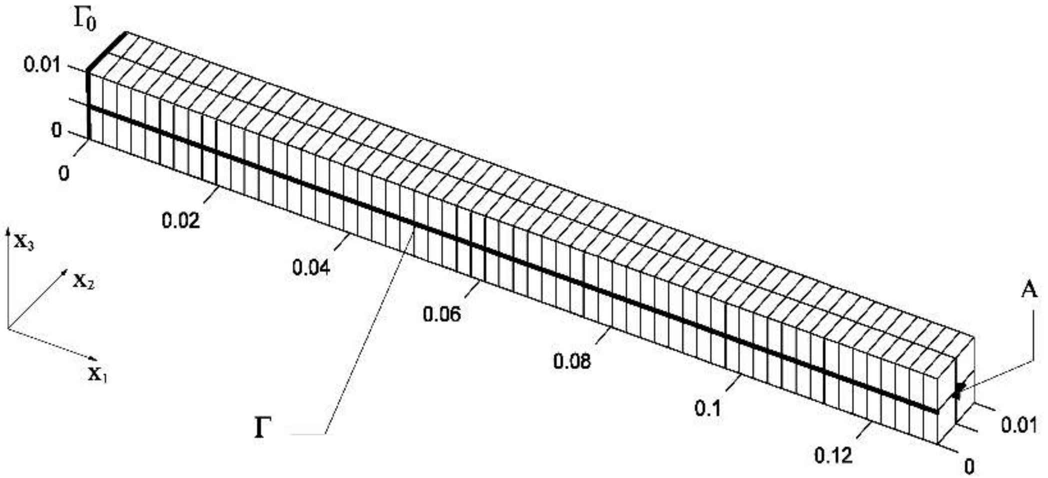
Algorithm for the maximum likelihood optimization

1. $[\mathbf{A}_0]$ with entries defined as independent uniform real random variables on $[-1\,,1]$.
2. $[\mathbf{B}_0] = [\mathbf{A}_0][\mathbf{A}_0]^T$
3. Cholesky decomposition of $[\mathbf{B}_0] = [\mathbf{L}]^T[\mathbf{L}]$
4. Columns of $[\mathbf{A}] = [\mathbf{L}]^{-T}[\mathbf{A}_0]$ are $\mathbf{A}_{\boldsymbol{\alpha}}$
   - $\sum_{\boldsymbol{\alpha},|\boldsymbol{\alpha}|=1}^{d} \mathbf{A}_{\boldsymbol{\alpha}}\mathbf{A}_{\boldsymbol{\alpha}}^T = [I_\mu]$
5. Estimate $p_{Q_j^{\nu,d}}(\{\mathbf{Q}(\theta_\ell^{\exp})\}_j; \mathbb{A}(\theta))$
6. Estimate $L(\mathbb{A}(\theta))$

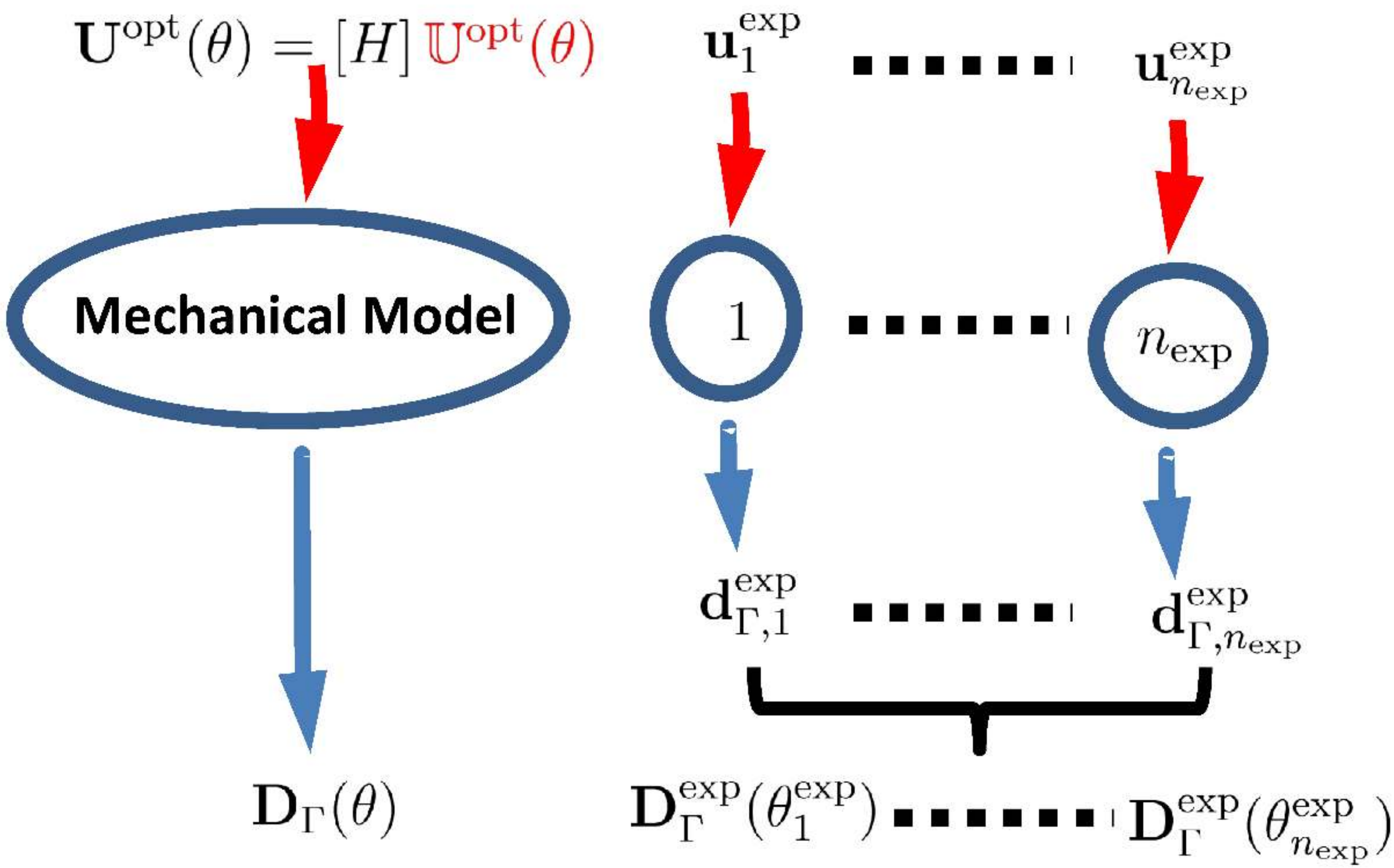
Outlines

1. The Young modulus is random: $\mathbf{U}^{\mathrm{ref}}(\mathbf{x}) = c_0\, g(c_1, c_2\, V(x_1))$
   - $g(\alpha,\theta) = F^{-1}_{\Gamma_\alpha}(F_\Theta(\theta))$
   - $V(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=1}^{3} H_{\boldsymbol{\alpha}}(\mathbf{Z})\,\sqrt{\gamma_{\boldsymbol{\alpha}}}\,\psi_{\boldsymbol{\alpha}}(\mathbf{x}/2)$
   - $C(\mathbf{x},\mathbf{x}') = \exp\left(-|x_1 - x_1'|/L\right)$

2. experimental data consist in $N_b = 60$ frequency response functions



Cost function for the first inverse problem



Modèle probabiliste de $\mathbb{U}^{\mathrm{opt}}$

$$\mathbb{U}^{\mathrm{opt}}(\theta) = \arg \min_{\mathbb{u} \in \mathbb{R}^{n \times N_i}} \left\{ \ell_{\mathrm{det}}(\mathbf{D}_\Gamma(\Theta); \mathbf{D}_\Gamma^{\exp}(\theta)) \right\}$$

Cost function for the first inverse problem

1. $$\mathbf{U}^{\text{opt}} = [H]\,\mathbb{U}^{\text{opt}}$$

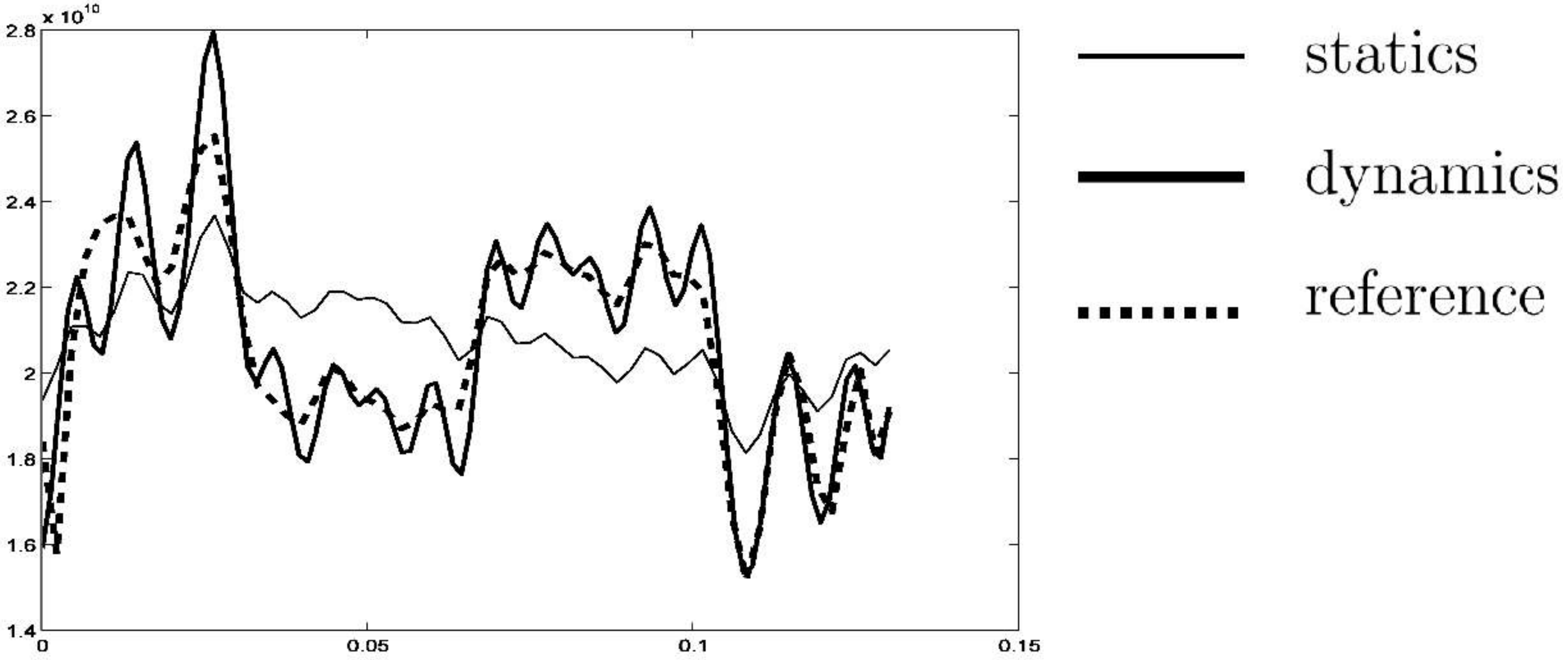
$$[A(\omega;\mathbf{U}^{\text{opt}})]\mathbf{D}(\omega) = \widehat{\mathbf{b}}(\omega)$$

$$\mathbf{D}_\Gamma(\omega) = r(\mathbf{D}(\omega))$$

2. $$\ell_{\text{det}}(\mathbf{D}_\Gamma(\theta);\mathbf{D}_\Gamma^{\text{exp}}(\theta)) = \sum_{k=1}^{N_{\text{band}}} \int_{B_k} \|\mathbf{D}_\Gamma(\omega;\theta) - \mathbf{D}_\Gamma^{\text{exp}}(\omega;\theta)\|^2 \, d\omega$$

3. $$\ell_{\text{det}}(\mathbf{D}_\Gamma(\theta);\mathbf{D}_\Gamma^{\text{exp}}(\theta)) = \|\mathbf{D}_\Gamma(0;\theta) - \mathbf{D}_\Gamma^{\text{exp}}(0;\theta)\|^2$$

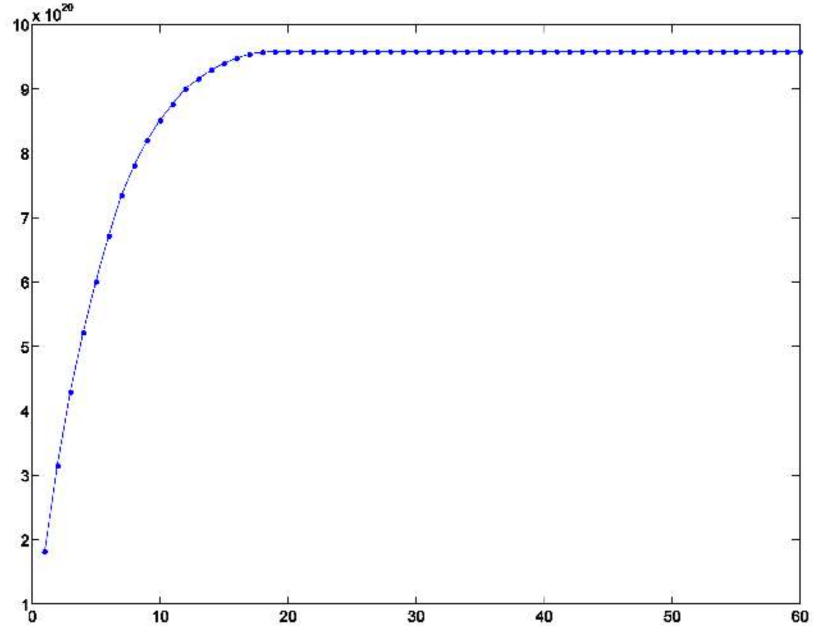
$$\mathbb{U}^{\text{opt}}(\theta) = \arg \min_{\mathbb{u}\in\mathbb{R}^{n\times N_i}} \left\{\ell_{\text{det}}(\mathbf{D}_\Gamma(\Theta);\mathbf{D}_\Gamma^{\text{exp}}(\theta))\right\}$$



Efficiency of the cost functions for idenifying the realization $\boldsymbol{U}^{\mathrm{opt}}(\theta_\ell^{\mathrm{exp}})$

- Better identification with dynamical measurements

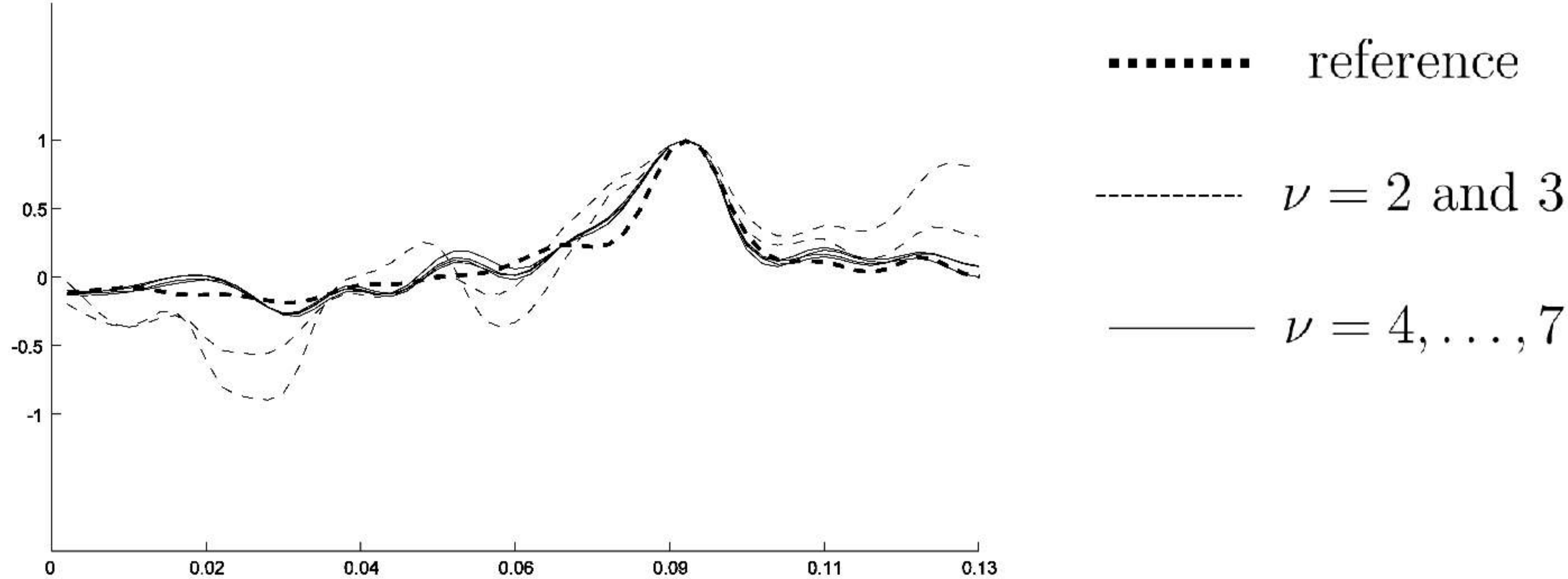
Convergence analysis



graph of $k \mapsto \sum_{k=1}^{\mu} \lambda_k$

- size of $\mathbf{Q}^{\nu,d}$ is $\mu = 20$
- 35 non gaussian coefficients were used in the reference probabilistic model
- $n_{\text{exp}} = 100$ experimental measurements are used

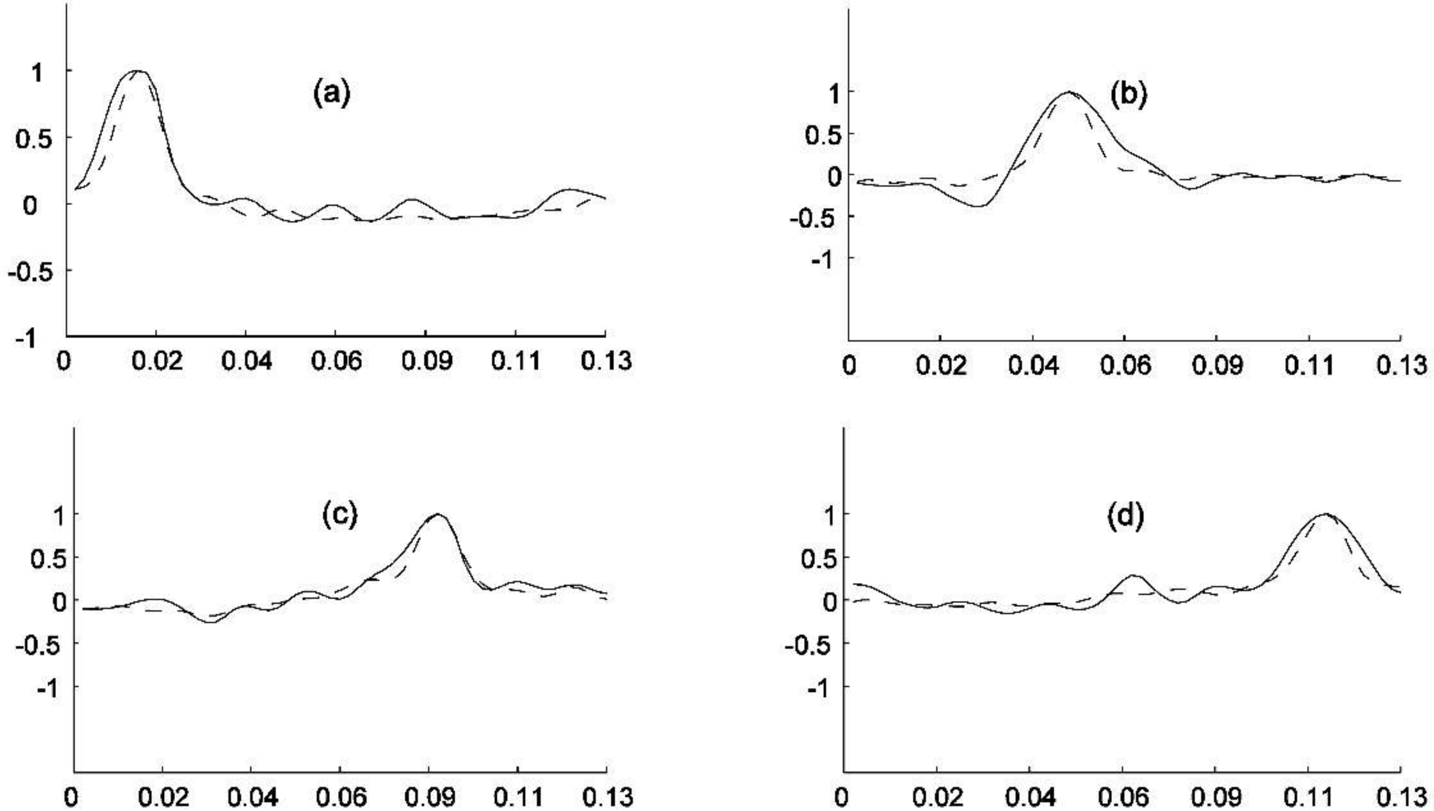
Convergence analysis with respect size of *W*



Correlation coefficient function

- PCE of $\mathbf{Q}^{\nu,d}$ is converged for $\nu \geq 4$ with $d = 5$
- size of $\mathbf{Z}$ in the reference model is 4

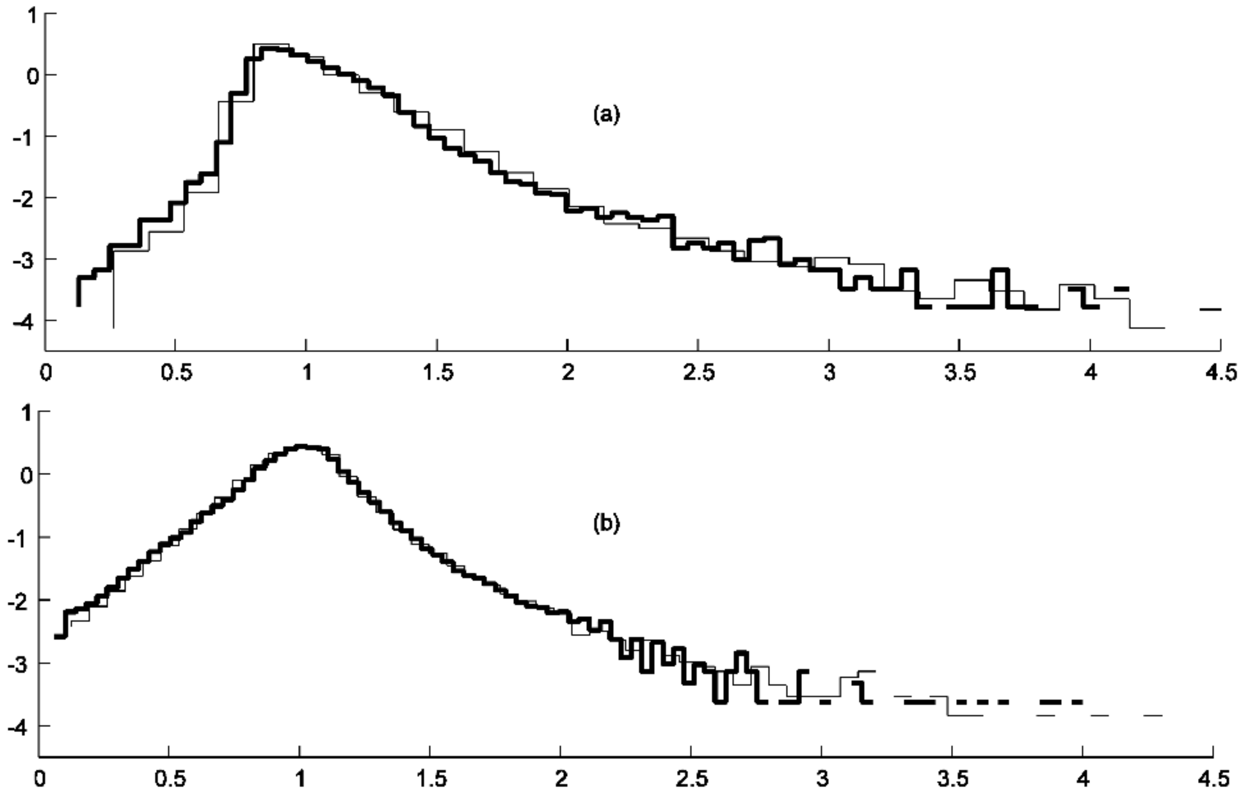
Results



Correlation coefficient function

- Results with $\mu = 20$ (size of $\mathbf{Q}^{\nu,d}$), with $\nu = 7$ (size of $\mathbf{W}^{\nu}$) and with $d = 5$ (degree of PCE)
- 15840 PCE coefficients !!

Results



Identified p.d.f. in logscale

- Results with $\mu = 20$ (size of $\mathbf{Q}^{\nu,d}$), with $\nu = 7$ (size of $\mathbf{W}^{\nu}$) and with $d = 5$ (degree of PCE)
- 15840 PCE coefficients !!

Conclusions

1. Stochastic inverse problem for a random field
2. Statistical reduction by solving an inverse problem
3. PCE of the vector of generalized coodinates of the random field
4. Likelihood optimization in high dimension
5. Estimate of the p.d.f. are carried out with realizations of the polynomial chaos
6. Computational loss of orthonormality of polynomial chaos

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Thank you for your attention