Adaptive stochastic coupling in the Arlequin method

C. Zaccardi* - L. Chamoin – R. Cottereau – H. Ben Dhia

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Context

- Structure with **local behavior modification** (cuts, holes, different processes...)

- **Multiscale**: close to singularities (load, BC...), close to / far from the defect...

Global homogeneous behavior with:
- **Strong variability** of parameters
- **Insufficient knowledge** of material parameters
- **Specific and complex physics** (cracking...)

Local Stochastic model superposed on a Deterministic one
Summary

1. Definition of the reference model
2. Reduced model with the Arlequin Method
   • The Arlequin method
   • Specificity of the coupling
   • Raw results on a simple case
3. Goal-oriented error estimation
   • Quantity of interest
   • Definition of the adjoint problem
   • Error estimation and adaptivity
4. Example of adaptive model with stochastic coupling
1. Definition of the reference model

Considering \((\Theta, \mathcal{F}, P)\) a complete probability space,

- **Equilibrium equation**
  \[
  \forall x \in \Omega, \quad \nabla \cdot (K(x, \theta) \nabla u(x, \theta)) = f(x)
  \]

- **Boundary conditions**
  \[
  \begin{align*}
  u &= 0 \text{ on } \Gamma_D \\
  K(x, \theta) \nabla u &= g(x) \text{ on } \Gamma_N
  \end{align*}
  \]
  a.s.

- **Stochastic material property**
  \[
  K(x, \theta) \in \mathcal{L}^2(\Theta, \mathcal{C}^0(\Omega))
  \]
  \[
  0 < K_{\min} \leq K(x, \theta) \leq K_{\max} < \infty, \quad \forall x \in \Omega \quad \text{a.s.}
  \]

In practice, the solution is unavailable.
2. Reduced model using the Arlequin method [Ben Dhia 1998-2008]

Key points

- model superposition
- volume coupling of the models
- distribution of the mechanical energy \( (\alpha_1(x), \alpha_2(x)) \)
Equilibrium equation [Cottereau 2010]

Find \( (u_d, u_s, \lambda) \in V_d \times W_s \times W_c \) such that:

\[
\begin{align*}
\mathcal{A}_d(u_d, v) + \mathcal{C}(\lambda, v) &= \ell_d(v), \quad \forall v \in V_d \\
\mathcal{A}_s(u_s, v) - \mathcal{C}(\lambda, v) &= \ell_s(v), \quad \forall v \in W_s \\
\mathcal{C}(\mu, \Pi u_d - u_s) &= 0, \quad \forall \mu \in W_c
\end{align*}
\]

Internal works

\[
a_d(u, v) = \int_{\Omega_1} \alpha_1(x) K_d(x) \nabla u \nabla v \, d\Omega
\]

External works

\[
\ell_d(v) = \int_{\Omega_1} \alpha_1(x) f v \, d\Omega
\]

\[
a_s(u, v) = \mathbb{E} \left[ \int_{\Omega_2} \alpha_2(x) K_s(x, \theta) \nabla u \nabla v \, d\Omega \right]
\]

\[
\ell_s(v) = \mathbb{E} \left[ \int_{\Omega_2} \alpha_2(x) f v \, d\Omega \right]
\]
Coupling operator and space [Cottereau 2011]

• Coupling operator  \( C : \mathcal{W}_c \times \mathcal{W}_c \to \mathbb{R} \)

\[
C(u, v) = \mathbb{E} \left[ \int_{\Omega_c} \kappa_0 uv + \kappa_1 \nabla u \nabla v \, d\Omega \right]
\]

• Coupling space

\[
\mathcal{W}_c = \{ v(x) + \theta \Pi_c(x) | v \in \mathcal{H}^1(\Omega_c), \theta \in \mathcal{L}^2(\Theta, \mathbb{R}), \int_{\Omega_c} v(x) \, d\Omega = 0 \}
\]

Random functions, perfectly spatially correlated

Deterministic

Random field with a spatially varying mean
Meanings of the coupling

\[ C(\mu, \Pi u_d - u_s) = 0, \forall \mu \in \mathcal{W}_c \]

\[ = C(\mathbb{E}[\mu], u_d - \mathbb{E}[u_s]) + \mathbb{E}\left[ \theta \int_{\Omega_c} (u_d - u_s) d\Omega \right] \]

With

\[ C'(u, v) = \int_{\Omega_c} \kappa_0 uv + \kappa_1 \nabla u \nabla v d\Omega \]

Equality between the mean of the stochastic field and the deterministic one

Average cancelling of the stochastic field variability
Simple application

• Monodimensional application (reference model)

\[
\begin{align*}
E[K(x, \theta)] &= 1 \\
L_{\text{correlation}} &= 0.01 \\
\sigma &= 0.2
\end{align*}
\]

• Stochastic distributed following a uniform law of bounds \([0.2294 ; 1.7706]\), with parameters:
Arlequin approximation

• Monodimensional application

\[ f = 1 \]

\[ u = 0 \]

\[ u = 1 \]

• Deterministic model described by: \( K_d(x) = 0.7537 \)

• \( K_s(x, \theta) \) distributed following a uniform law with parameters:

\[
\begin{align*}
E[K_s(x, \theta)] &= 1 \\
L_{\text{correlation}} &= 0.01 \\
\sigma &= 0.2
\end{align*}
\]

Remark:
To ensure the physical meaning of the coupling

\[ K_d^{-1} = E[K_s^{-1}] \]
Solution: gradient of the displacement

Dashed black lines: mean and 90% confidence interval with a full stochastic monomodel
Solution: gradient of the displacement

Continued black lines: deterministic solution, and mean of the stochastic one
Yellow zone: 90% confidence interval (representation of the fluctuation)
Dashed black lines: mean and 90% confidence interval with a full stochastic monomodel
3. Error estimation

**Interest:**
- Control the quality of the solution
- Drive an adaptive model
Goal-oriented error estimation [Prudhomme 1999] [Oden 2001]

- Local quantity of interest: $q(u)$
  - Mean of the displacement of a point
  - Local average of the standard deviation of the stress field

- Use of global error estimation with Extraction techniques:
  Example with the mathematical expectation of a displacement

  \[ q(u) = E[u(x = x_m)] \]

  \[ q(u) = E \left[ \int_\Omega \delta(x - x_m) u \, d\Omega \right] \]

- Related error

  \[ \eta = q(u^{ex}) - q(u^0) \]

- Parameters: $L_s, h_d, h_s$
Definition of the adjoint problem

- Primal reference problem (1)

Find $u \in V$ such that:

$$a(u, v) = \ell(v), \quad \forall v \in V$$

- Adjoint problem if $q$ is linear:

Find $p \in V$ such that:

$$a(v, p) = q(v), \quad \forall v \in V$$

$$q(u) = E \left[ \int_{\Omega} \delta(x - x_m) u \, d\Omega \right]$$

- The adjoint problem is still defined on the reference model.
Approximation of the adjoint problem

Using the Arlequin method

Primal:

Find \((u_d, u_s, \lambda) \in \mathcal{V}_d \times \mathcal{W}_s \times \mathcal{W}_c\) such that:

\[
\begin{align*}
    a_d(u_d, v) + C(\lambda, v) &= \ell_d(v), \quad \forall v \in \mathcal{V}_d \\
    a_s(u_s, v) - C(\lambda, v) &= \ell_s(v), \quad \forall v \in \mathcal{W}_s \\
    C(\mu, \Pi u_d - u_s) &= 0, \quad \forall \mu \in \mathcal{W}_c
\end{align*}
\]

Adjoint with \(q\) linear:

Find \((\tilde{p}_{u_d}, \tilde{p}_{u_s}, \tilde{p}_\lambda) \in \tilde{\mathcal{V}}_d \times \tilde{\mathcal{W}}_s \times \tilde{\mathcal{W}}_c\) such that:

\[
\begin{align*}
    a_d(v, \tilde{p}_{u_d}) + C(v, \tilde{p}_\lambda) &= q_d(v), \quad \forall v \in \tilde{\mathcal{V}}_d \\
    a_s(v, \tilde{p}_{u_s}) - C(v, \tilde{p}_\lambda) &= q_s(v), \quad \forall v \in \tilde{\mathcal{W}}_s \\
    C(\Pi \tilde{p}_{u_d} - \tilde{p}_{u_s}, \mu) &= 0, \quad \forall \mu \in \tilde{\mathcal{W}}_c
\end{align*}
\]

Quality control by estimation of the global error
Error estimation

- Estimation of the error for linear quantity of interest:

\[ \eta = q(u^{ex}) - q(u^0) = R(u, p) \approx R(u, \tilde{p}) \]

Where the residual \( R : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \) is defined by:

\[ R(u, v) = l(v) - a(u, v) \]

and where \( u \) and \( \tilde{p} \) are projections of \( (u_d, u_s, \lambda) \) and of \( (\tilde{p}_u, \tilde{p}_u, \tilde{p}_\lambda) \) in \( \mathcal{V} \) respectively.

For instance:

\[ u = \begin{cases} u_d & \text{in } \Omega_d \setminus \Omega_s \\ u_s & \text{in } \Omega_s \end{cases} \]

[Prudhomme 2008]
Error sources

• Several error sources:
  ✓ Modeling error (Arlequin and Stochastic homogenization)
  ✓ Spatial discretization error (FEM)
  ✓ Stochastic discretization (Truncation of the Monte Carlo method)

• Introducing intermediate models
  ➢ Continuum deterministic-stochastic Arlequin model (arl)
  ➢ Arlequin model only discretized in space (arl$^h$)
  ➢ Arlequin model discretized in space and using Monte Carlo (arl$^{h\theta}$)

• Decomposition of the error:

$$\eta = q(u^{ex}) - q(u^0)$$
$$= (q(u^{ex}) - q(u^{arl})) + (q(u^{arl}) - q(u^h)) + (q(u^h) - q(u^0))$$

$\eta^m$ modeling error
$\eta^h$ discretization error
$\eta^{\theta}$ stochastic error
4. Example of adaptive coupling

- Approximated model (primal)

\[ f = 1 \]
\[ u = 0 \]
\[ \Omega_d \]
\[ \Omega_s \]
\[ K_s(x, \theta) \]
\[ \Omega_c \]
\[ K_d(x) \]
\[ L_c \]
\[ L_s \]

Numerical model: discretization by FEM and Monte Carlo techniques

- Adjoint problem

\[ f = 1 \]
\[ u = 0 \]
\[ \Omega_d^h \]
\[ \Omega_s^h \]
\[ K_s(x, \theta_i) \]
\[ \Omega_c^h \]
\[ L_c^h \]
\[ L_s^h \]
\[ K_d(x)u = 0 \]
Evolution of the modeling error for different $L_s$

- Reference model
- Approximated model
- Error estimation
- Example

Graph showing the evolution of the modeling error for different $L_s$.
Evolution of the discretization error for different $h_d, h_s$
Remark about the example

Errors on the example are relatively small
Conclusion and extensions

Outcomes
✓ Efficient deterministic - stochastic coupling
✓ Goal oriented error estimation with a linear quantity of interest
✓ Error sources

Outlines
➢ Other intermediate problems (sto-sto,semi-discretized…)
➢ Importance of the projection into the reference admissible space
➢ Extension to the coupling of particular stochastic model with a deterministic continuum one
Thank you for your attention


