Sensitivity analysis method for failure probability

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Abstract

Sensitivity analysis of a numerical model, for instance simulating physical phenomena, is a tool used to explore, understand and (partially) validate computer codes. It aims at explaining the outputs regarding the input uncertainties [1]. This communication proposes a sensitivity index, based upon the modification of the probability density function (pdf) of the random inputs, when the quantity of interest is a failure probability (probability that a model output exceeds a given threshold). This work is an extension of a submitted article [4].

1. Model, aim of the study

- We study a **deterministic numerical model** denoted $G : \mathbb{R}^d \to \mathbb{R}$
- The **uncertain inputs** are denoted $\mathbf{X}$ which is a d-dimensional random variable - of joint pdf f.
- We consider independent inputs of marginal densities f_i for $i = 1..d$.
- The **uncertain output** is denoted $Z = G(\mathbf{X})$ and is a random variable.
- We will have a specific interest for binary output (reliability context). We consider the event $G(\mathbf{x}) < 0$ (system failure) and the complementary event $G(\mathbf{x}) \geq 0$ (system safe mode).
- The quantity of interest is the system **failure probability**:

$$P = \int \mathbf{1}_{\{G(\mathbf{x})<0\}} f(\mathbf{x}) d\mathbf{x}. \tag{1}$$

- The aim of this work is the quantification of the **influence** of each variable X_i on this probability.
- In most real cases, the **family** of the distribution of an input is given by the physic wherehas the **parameters** of such a distribution are data-driven.
- The question guiding us throughout this presentation is:

"what would be the impact on P of a parameterization error?"

2. Methodology of input perturbation

- Given an input variable X_i with pdf f_i, let us call $X_{i\delta} \sim f_{i\delta}$ the corresponding **perturbed** random input.
- This **perturbed** input takes the place of the real random input X_i, in a sense of modelling error : what if the real distribution were $X_{i\delta}$ instead of X_i?
- More precisely, we suggest to define a perturbed input density $f_{i\delta}$ as the closest distribution to the original f_i in the **entropy sense** and under some **constraints of perturbation**.
- Recall that between two pdf p and q we have:

$$KL(p, q) = \int_{-\infty}^{+\infty} p(y) \log \frac{p(y)}{q(y)} dy \text{ if } \log \frac{p(y)}{q(y)} \in L^1(p(y)dy). \tag{2}$$

- Let $i = 1, \cdots, d$, the constraints express as follows:

$$\int g_k(x_i) f_{\text{mod}}(x_i) dx_i = \delta_{k,i} \quad (k = 1 \cdots K). \tag{3}$$

- For $k = 1, \cdots, K$, g_k are given functions and $\delta_{k,i}$ are given reals. These quantities will lead to a **perturbation** of the original density.
- The **modified density** $f_{i\delta}$ considered in our work is:

$$f_{i\delta} = \underset{f_{\text{mod}} \mid (3) \text{ holds}}{\text{argmin}} KL(f_{\text{mod}}, f_i) \tag{4}$$

and the result takes an explicit form [6] given in the following equations.

- Let us define, for $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_K)^T \in \mathbb{R}^K$,

$$\psi_i(\boldsymbol{\lambda}) = \log \int f_i(x) \exp \left[\sum_{k=1}^{K} \lambda_k g_k(x) \right] dx. \tag{5}$$

- Then, there exists a unique $\boldsymbol{\lambda}^*$ such that the solution of the minimisation problem (4) is:

$$f_{i\delta}(x_i) = f_i(x_i) \exp \left[\sum_{k=1}^{K} \lambda_k^* g_k(x_i) - \psi_i(\boldsymbol{\lambda}^*) \right]. \tag{6}$$

3. Examples of perturbations

Mean shifting The first moment is often used to parametrize a distribution. Thus the first perturbation presented here is a **mean shift**, that is expressed with a single constraint:

$$\int x_i f_{\text{mod}}(x_i) dx_i = \delta_i. \tag{7}$$

Variance shifting In some cases, the expectation of an input may not be the main source of uncertainty. One might be interested in perturbating the second moment of an input. This case may be treated considering a couple of constraints. The perturbation presented is a **variance shift**, therefore the set of constraints is:

$$\begin{cases} \int x_i f_{\text{mod}}(x_i) dx_i = \mathbb{E}[X_i], \\ \int x_i^2 f_{\text{mod}}(x_i) dx_i = \delta_i + \mathbb{E}[X_i]^2. \end{cases} \tag{8}$$

Quantile shifting As far as we noticed, in most cases the values of the input leading to the failure event comes from the tails of the input distributions. We therefore propose a **quantile perturbation**. Denoting q_r the reference quantile; e.g. the value so that: $\int_{-\infty}^{q_r} f(x)dx = r$, one can express a quantile perturbation:

$$\int 1_{]-\infty:q_r]}(x_i) f_{\text{mod}}(x_i) dx_i = \delta_i \tag{9}$$

meaning that f_{mod} is a density such that its δ_i-quantile is q_r.

4. Sensitivity index

- According to the **pertubations** defined in the previous section, the failure probability becomes:

$$P_{i\delta} = \int \mathbf{1}_{\{G(\mathbf{x})<0\}} \frac{f_{i\delta}(x_i)}{f_i(x_i)} f(\mathbf{x}) d\mathbf{x}. \tag{10}$$

- One can define the **sensitivity index**:

$$S_{i\delta} = \left[\frac{P_{i\delta}}{P} - 1 \right] \mathbf{1}_{\{P_{i\delta} \geq P\}} + \left[1 - \frac{P}{P_{i\delta}} \right] \mathbf{1}_{\{P_{i\delta} < P\}}. \tag{11}$$

  - $S_{i\delta} = 0$ if $P_{i\delta} = P$, as expected if X_i is a non-influential variable or if δ expresses a negligible perturbation.
  - The sign of $S_{i\delta}$ indicates how the perturbation impacts the failure probability qualitatively.

- These sensitivity indices can be estimated using **the sole set of simulations** that has already been used to compute the failure probability P, thus limiting the number of calls to the numerical model. Under mild **support constraints**, one can consistently estimate $P_{i\delta}$ by:

$$\hat{P}_{i\delta N} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{\{G(\mathbf{x}^n)<0\}} \frac{f_{i\delta}(x_i^n)}{f_i(x_i^n)}. \tag{12}$$

where $(\mathbf{x}_i^1, \ldots, \mathbf{x}_i^N)$ is a Monte-Carlo sample. This property holds in the more general case when P is originally estimated by importance sampling rather than simple Monte Carlo, which is more appealing in contexts when G is time-consuming [2,3]. **Asymptotical properties** of the indices are derived [4], including a CLT for $\hat{S}_{i\delta N}$, the plug-in estimator of $S_{i\delta}$. One has indeed:

$$\sqrt{N} \left[\hat{S}_{i\delta N} - S_{i\delta} \right] \xrightarrow[N \to \infty]{\mathcal{L}} \mathcal{N}\left(0, d^T \Sigma d\right) \tag{13}$$

where d is the derivative in $(P, P_{i\delta})$ of the continuous function $s(x, y) = \left[\frac{y}{x} - 1\right] \mathbf{1}_{y \geq x} + \left[1 - \frac{x}{y}\right] \mathbf{1}_{y<x}$, $\hat{S}_{i\delta N}$.

5. Numerical applications

Hyperplane test case The failure function is defined as:

$$G(\mathbf{X}) = k - \sum_{i=1}^{4} a_i X_i$$

with $f_{X_i} \sim \mathcal{N}(0, 1)$ for $i = 1, .., 4$, $k = 16$ and $\mathbf{a} = (1, -6, 4, 0)$. One has $P \simeq 0.014$.

Variable	X_1	X_2	X_3	X_4
Importance factor	0.018	0.679	0.302	0

Group index	S_1	S_2	S_3	S_4	S_{T1}	S_{T2}	S_{T3}	S_{T4}
Sobol' index of $\mathbf{1}_{\{G(\mathbf{x})<0\}}$	0.0017	0.2575	0.0544	0	0.1984	0.9397	0.7256	0

Table 1: *FORM Importance factors and Sobol' indices for hyperplane function*

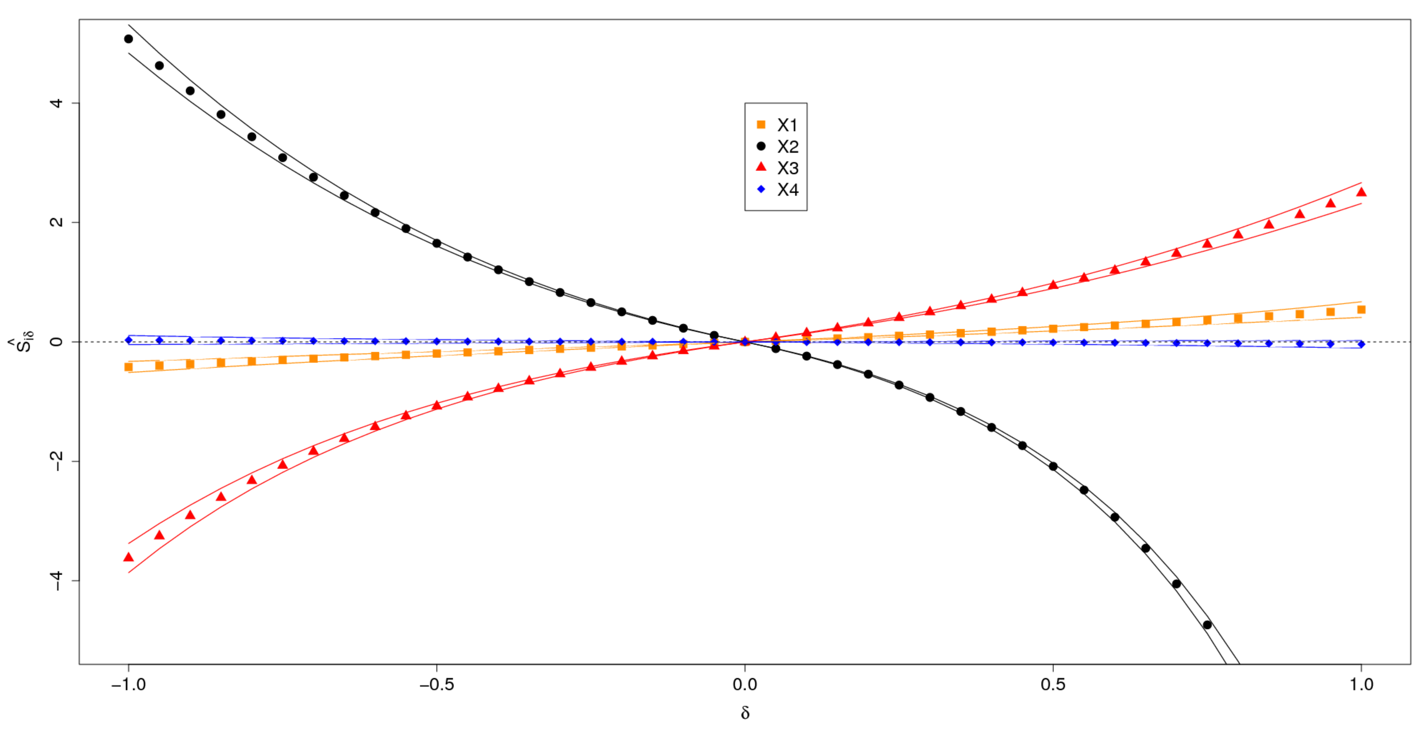

Figure 1: *Estimated indices $\widehat{S_{i\delta}}$ for hyperplane function with a mean shifting*

Ishigami test case A modified version of the Ishigami function will be considered:

$$G(\mathbf{X}) = \sin(X_1) + 7\sin(X_2)^2 + 0.1X_3^4 \sin(X_1) + 7$$

where $X_i \sim \mathcal{U}[-\pi, \pi], i = 1, \ldots, 3$.. The failure probability here is roughly $P \simeq 0.006$.

Variable	X_1	X_2	X_3
Importance factor	$1e^{-17}$	1	0

Group index	S_1	S_2	S_3	S_{T1}	S_{T2}	S_{T3}
Sobol' index of $\mathbf{1}_{\{G(\mathbf{x})<0\}}$	0.0234	0.0099	0.0667	0.8158	6758	0.9299

Table 2: *FORM Importance factors and Sobol' indices for the thresholded Ishigami function*

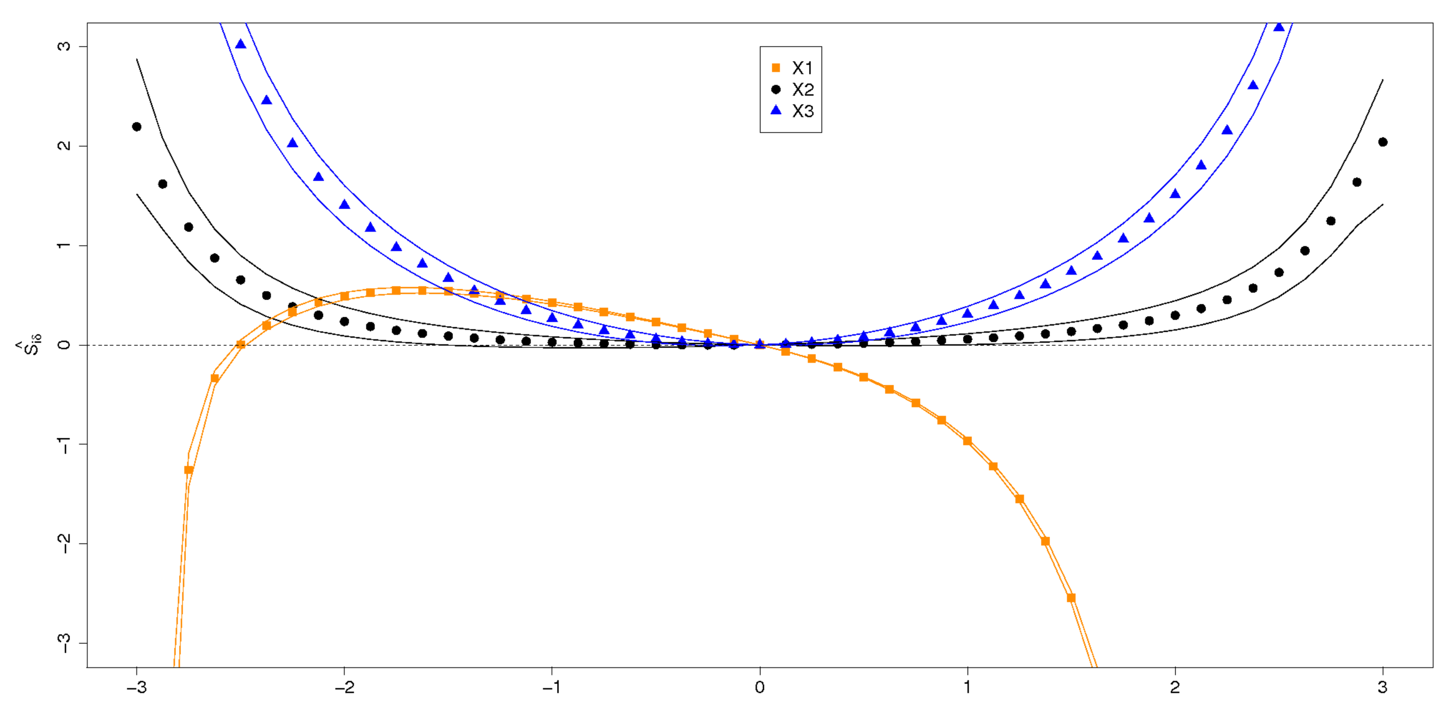

Figure 2: *Estimated indices $\widehat{S_{i\delta}}$ for thresholded Ishigami function with a mean shifting*

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