ROUGH PATHS THEORY APPLIED TO COMPUTATION OF SENSITIVITIES

Nicolas MARIE
Université Paris-Ouest - Laboratoire Modal'X
nmarie@u-paris10.fr

Preliminaries

- $d \in \mathbb{N}^*$ and $T > 0$.
- B^H a d-dimensional fractional Brownian motion (FBM) with Hurst parameter $H > 1/4$.
- $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to GL_d(\mathbb{R})$ of class $C^{[1/H]+1}$, bounded, with bounded derivatives and σ^{-1} bounded.
- X the solution on $[0, T]$, in rough paths sense, of the stochastic differential equation (SDE)

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB^H_t \tag{1}$$

with a deterministic initial condition $x \in \mathbb{R}^d$.

The problem

Existence and computation of

$$\Delta^{x,\cdot} := \partial_x \mathbb{E}\left[F(X_T)\right] \text{ and } \mathcal{V}^{\sigma,\cdot} := \partial_\sigma \mathbb{E}\left[F(X_T)\right]$$

for $F : \mathbb{R}^d \to \mathbb{R}$ piecewise continuous, satisfying the following assumption :

Assumption 1. There exists $C > 0$ and $N \in \mathbb{N}^*$ such that : $|F(a)| \leqslant C(1 + \|a\|)^N$; $\forall a \in \mathbb{R}^d$.

Notations

- $\mathcal{H}^1$ the Cameron-Martin's space of B^H.
- $\mathcal{H}$ the reproducing kernel Hilbert space of B^H.
- I_H the canonical isometry from $\mathcal{H}$ into $\mathcal{H}^1$.
- $\mathbf{D}$ the Malliavin derivative operator for B^H and δ the associated divergence.
- $\delta_{1/2}$ the divergence for the Brownian motion associated to B^H in its representation as a Volterra process.
- $J_{.\leftarrow 0}$ the Jacobian matrix of $x \mapsto X$.

Theorem 1

- If F is continuously differentiable and, F and DF satisfy Assumption 1, $\Delta^{x,\hat{x}}$ and $\mathcal{V}^{\sigma,\hat{\sigma}}$ exist. Moreover,

$$\Delta^{x,\hat{x}} = \mathbb{E}\left[\langle \mathbf{D}(F \circ X_T), I_H^{-1}(h^{x,\hat{x}})\rangle_{\mathcal{H}}\right] \text{ and } \mathcal{V}^{\sigma,\hat{\sigma}} = \mathbb{E}\left[\langle \mathbf{D}(F \circ X_T), I_H^{-1}(h^{\sigma,\hat{\sigma}})\rangle_{\mathcal{H}}\right]$$

where,

$$h^{x,\hat{x}} := \frac{1}{T}\int_0^{\cdot} \sigma^{-1}(X_s)\, J_{s\leftarrow 0}\hat{x}\,ds \text{ and } h^{\sigma,\hat{\sigma}} := \frac{1}{T}\int_0^{\cdot} \sigma^{-1}(X_s)\, J_{s\leftarrow T}\langle \partial_\sigma X_T, \hat{\sigma}\rangle ds.$$

- If F is piecewise continuous and satisfies Assumption 1, for $H > 1/2$; $\Delta^{x,\hat{x}}$ and $\mathcal{V}^{\sigma,\hat{\sigma}}$ exist. Moreover,

$$\Delta^{x,\hat{x}} = \mathbb{E}\left[F(X_T)\delta\left[I_H^{-1}(h^{x,\hat{x}})\right]\right] \text{ and } \mathcal{V}^{\sigma,\hat{\sigma}} = \mathbb{E}\left[F(X_T)\delta\left[I_H^{-1}(h^{\sigma,\hat{\sigma}})\right]\right]$$

with, for $h = h^{x,\hat{x}}$ or $h^{\sigma,\hat{\sigma}}$:

$$\delta\left[I_H^{-1}(h)\right] = \frac{1}{\Gamma(3/2 - H)}\int_0^T \left[t^{H-1/2}\frac{d}{dt}\int_0^t (t-s)^{1/2-H} s^{1/2-H}\dot{h}_s ds\right]\delta_{1/2}B_t.$$

An application in finance

Consider a financial market consisting of d risky assets and S the associated prices process :

$$\begin{cases} S_t := \kappa(X_t) \\ dX_t = \mu(X_t)\,dt + \sigma(Y_t)\,dB^1_t \\ dY_t = \vartheta(Y_t)\,dB^2_t \end{cases}$$

with :

- B^1 and B^2 two independent d-dimensional FBM.
- $\kappa : \mathbb{R}^d \to \mathbb{R}^d_+$ such that $F \circ \kappa$ satisfies Assumption 1.
- $\vartheta : \mathbb{R}^d \to GL_d(\mathbb{R})$ satisfying the same assumptions than σ.

Theorem 1 allows to compute

$$\langle \partial_\vartheta \mathbb{E}\left[F(S_T)\right] ; \hat{\vartheta}\rangle \text{ as } \langle \partial_V \mathbb{E}[(F \circ \kappa \circ p_{\mathbb{R}^d_1})(Z_T)]; \hat{V}\rangle$$

with $Z := (X, Y)$ satisfying :

$$dZ_t = \begin{bmatrix} (\mu \circ p_{\mathbb{R}^d_1})(Z_t) \\ 0 \end{bmatrix} dt + V(Z_t)\,dW_t$$

for $W := (B^1, B^2)$,

$$V(x,y) := \begin{bmatrix} \sigma(y) & 0 \\ 0 & \vartheta(y) \end{bmatrix} \text{ and } \hat{V}(x,y) := \begin{bmatrix} 0 & 0 \\ 0 & \hat{\vartheta}(y) \end{bmatrix} \; ; \forall (x,y) \in \mathbb{R}^d_1 \oplus \mathbb{R}^d_2.$$

Simulations

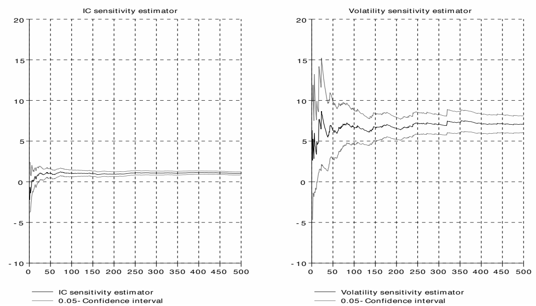
Consider X^n, Y^n and Z^n ($n \in \mathbb{N}^*$) the forward step-n Euler schemes for X, $\partial_x X$ and $\partial_\sigma X.\hat{\sigma}$ in dimension 1.

Each Euler scheme converges uniformly in $L^r(\Omega)$ with rate $n^{r(1-2\alpha)}$; $1/2 < \alpha < H$ et $r \geqslant 1$.

Then, for n sufficiently high, the law of large numbers allows to approximate Δ^x and $\mathcal{V}^{\sigma,\hat{\sigma}}$ by :

$$\Theta^n_m(x) := \frac{1}{m}\sum_{k=1}^m \dot{F}[X^n_k(T)]Y^n_k(T) \text{ and } \Theta^n_m(\sigma,\hat{\sigma}) := \frac{1}{m}\sum_{k=1}^m \dot{F}[X^n_k(T)]Z^n_k(T).$$

The central limit theorem provides confidence intervals.



References :

[1] T. Cass, C. Litterer and T. Lyons, *Integrability Estimates for Gaussian Rough Differential Equations*, arXiv: 1104.1813v4, 2011.
[2] P. Friz and N. Victoir, *Multidimensional Stochastic Processes as Rough Paths : Theory and Applications*, Cambridge Studied in Applied Mathematics, 120, Cambridge University Press, Cambridge, 2010.
[3] E. Fournié, J-M. Larsy, J. Lebuchoux, P-L. Lions and N. Touzi, *Applications of Malliavin Calculus to Monte-Carlo Methods in Finance*, Finance Stochast., 3:391–412, 1999.
[4] E. Gobet and R. Münos, *Sensitivity Analysis using Itô-Malliavin Calculus and Martingales, and Application to Stochastic Optimal Control*, SIAM J. Control Optim., 43(5):1676–1713, 2005.
[5] N. Marie, *Sensitivities via Rough Paths*, arXiv:1108.0852v8, 2011.
[6] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer-Verlag, New York, 2nd edition, 2006.