Comparing conservative estimations of failure probabilities using sequential designs of experiments in monotone frameworks

V. Moutoussamy[1,2]

P. Rochet[2] **T. Klein**[2] **N. Bousquet**[1]

1-EDF R&D, Chatou, France; 2-Université Paul Sabatier, Toulouse

`vincent.moutoussamy@edf.fr`

Abstract

A key point of structural reliability studies is to estimate the probability of an undesirable event. This estimation is made possible by using a numerical code that mimics the physical behavior of the studied phenomenon. The events considered are usually rare, accuring with a low probability. These constraints forbid in practice to use crude Monte Carlo methods. Variance reduction methods must be carried out to provide usable estimations in due time. This problem drives to the development and adaptation to methods to reduce the number of calls of the code.

1. Industrial Context

- Let $G:\mathbb{R}^d\to\mathbb{R}$ be a **deterministic numerical model**
- Uncertain inputs are represented by a **random vector** $\mathbf{X}$
- An indesirable event is described as $G(\mathbf{X})<0$
- The quantity of interest is

$$p=\mathbb{P}(G(\mathbf{X})\le 0) \tag{1}$$

2. Monotonicity

- A function G is said **monotonic** if

$$\forall \mathbf{x}\in\mathbb{U}, \forall i\in\{1,\dots,d\}, \forall \epsilon\in\mathbb{R}^*_+, \exists s_i\in\{-1,1\}, G(x_1,\dots,x_i+s_i\epsilon,\dots,x_d)\le G(x_1,\dots,x_d) \tag{2}$$

- Unless lost of generality one assume $\mathbb{U}=[0,1]^d$ and G is **increasing** in all inputs. Let X_i and define F_i the cumulative density function of $\mathbf{X}_i$. One transform the input as

$$\begin{cases}\mathbf{X}_i\leftarrow F_i(\mathbf{X}_i) & \text{if } G \text{ is increasing in } \mathbf{X}_i\\ \mathbf{X}_i\leftarrow 1-F_i(\mathbf{X}_i) & \text{if } G \text{ is decreasing in } \mathbf{X}_i\end{cases} \tag{3}$$

- Let $\bar{\mathbf{x}}_n=\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$ obtained from **any method of simulation** and evaluated by G
- Considering

$$\Xi_n^- = \{\mathbf{x}\in\bar{\mathbf{x}}_n : G(\mathbf{x})\le 0\} \tag{4}$$
$$\Xi_n^+ = \{\mathbf{x}\in\bar{\mathbf{x}}_n : G(\mathbf{x})> 0\} \tag{5}$$
$$\mathbb{U}_n^- = \{\mathbf{x}\in\mathbb{U} : \exists\mathbf{y}\in\Xi_n^-; \mathbf{x}\preceq\mathbf{y}\} \tag{6}$$
$$\mathbb{U}_n^+ = \{\mathbf{x}\in\mathbb{U} : \exists\mathbf{y}\in\Xi_n^+; \mathbf{x}\succeq\mathbf{y}\}, \tag{7}$$

- Two **exact and deterministic** bounds for p can be obtained for all $n\ge 0$:

$$\mathbb{P}(\mathbf{U}\in\mathbb{U}_n^-)=p_n^-\le p\le p_n^+=1-\mathbb{P}(\mathbf{U}\in\mathbb{U}_n^+) \tag{8}$$

with $\mathbf{U}$ uniformly distributed on $\mathbb{U}$

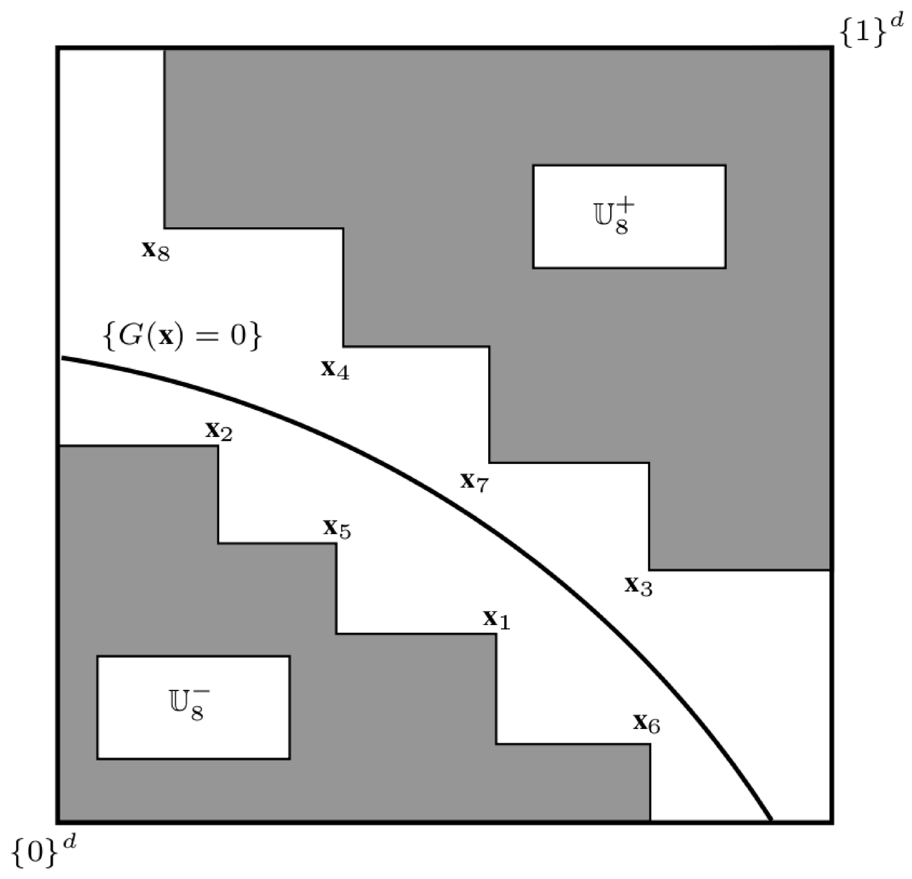

Figure 1: *In dimension 2, $\Xi_8^-=\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_5,\mathbf{x}_6\}$ and $\Xi_8^+=\{\mathbf{x}_3,\mathbf{x}_4,\mathbf{x}_7,\mathbf{x}_8\}$.*

3. Sequential Importance Sampling

- Assume that at step n of the exploration of input space $\mathbb{U}$, the next point $\mathbf{x}_n$ of the design is sampled from the importance distribution

$$\mathbf{x}_n \sim f_{n-1}\equiv\mathcal{N}_d(\mathbf{x}^*_{n-1},\sigma^2 I_d)\mathbb{1}_{\{\mathbf{x}\in\mathbb{U}_{n-1}\}} \tag{9}$$

- The idea is to choose $\mathbf{x}^*_{n-1}$ which maximize a criterion C such that $\mathbf{x}^*_{n-1}$ is near of Γ
- Denote

$$p_{n+1}^\pm(\mathbf{x}) = \mathbb{P}(\mathbf{U}\in\mathbb{U}_{n+1}^\pm(\mathbf{x})) \tag{10}$$

the contribution of $\mathbf{x}$ for the reduction of the bounds. Where

$$\mathbb{U}_{n+1}^-(\mathbf{x})=\{\mathbf{z}\in\mathbb{U}:\exists\mathbf{y}\in(\Xi_n^-\cup\mathbf{x});\mathbf{z}\preceq\mathbf{y}\},\quad \mathbb{U}_{n+1}^+(\mathbf{x})=\{\mathbf{z}\in\mathbb{U}:\exists\mathbf{y}\in(\Xi_n^+\cup\mathbf{x});\mathbf{z}\succeq\mathbf{y}\} \tag{11}$$

- Two classes of methods are proposed, the first one based on **geometrical** criterion and the second one based on **classification** tools
- The first criterion is the **volumetric-maximin (V-Maximin)** and is describe as follow

$$C(\mathbf{x})=\min(p_{n+1}^-(\mathbf{x})-p_n^-, p_n^+-p_{n+1}^+(\mathbf{x})) \tag{12}$$

- An alternative criterion called **quick-maximin (Q-Maximin)** is proposed

$$\tilde{C}(\mathbf{x}) = \min(c_{n+1}^-(\mathbf{x}),c_{n+1}^+(\mathbf{x})) \tag{13}$$

where

$$\mathbf{x}\in\bar{\mathbf{y}}_n = (\mathbf{y}_1,\dots,\mathbf{y}_n)\sim Uniform(\mathbb{U}^n) \tag{14}$$
$$c_{n+1}^-(\mathbf{x})=\#\{\mathbf{y}\in\bar{\mathbf{y}}_n:\mathbf{y}\preceq\mathbf{x}\}\quad,\quad c_{n+1}^+(\mathbf{x})=\#\{\mathbf{y}\in\bar{\mathbf{y}}_n:\mathbf{y}\succeq\mathbf{x}\} \tag{15}$$

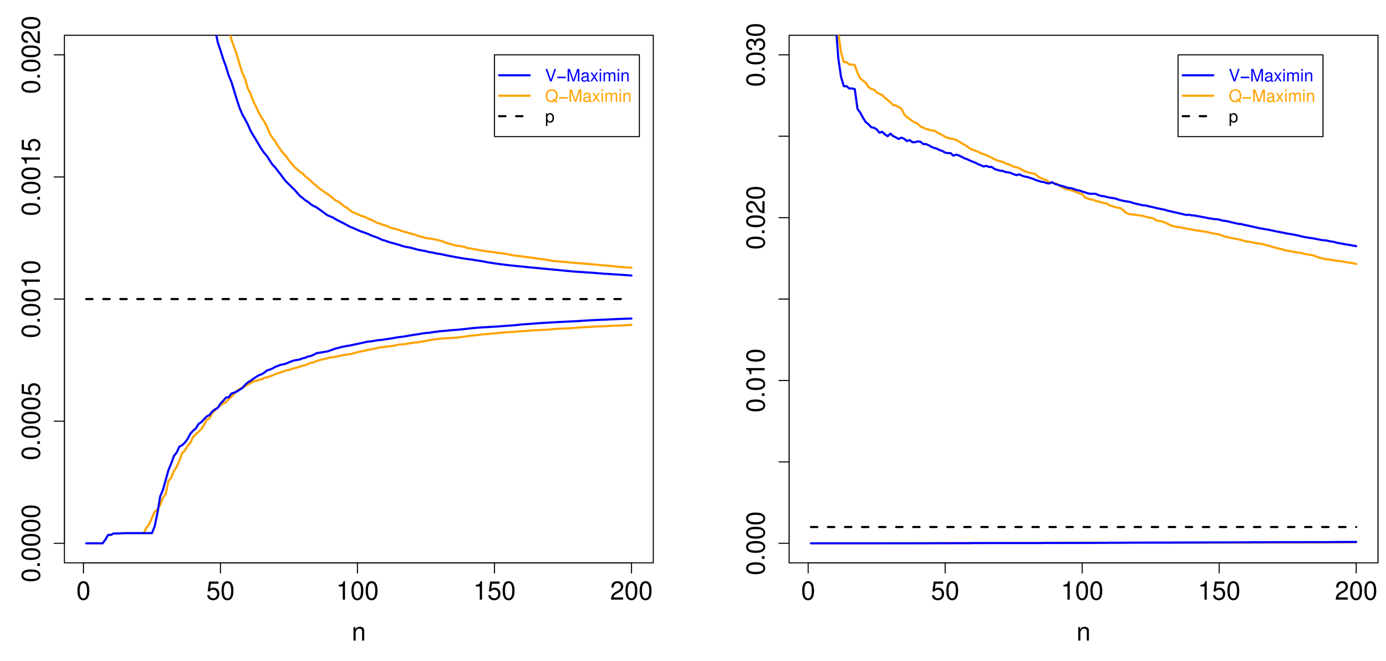

Figure 2: *Comparison of methods V-Maximin and Q-Maximin*

- The second approach is based on classification tools. The problem to class a new point is a problem of **binary classification**, which can be solved using monotonic neural networks. One proposes three criteria:

$$C_1(\mathbf{x}) = [p_n^+-p_{n+1}^+(\mathbf{x})]\pi_1(\mathbf{x})+[p_{n+1}^-(\mathbf{x})-p_n^-]\pi_{-1}(\mathbf{x}) \tag{16}$$
$$C_2(\mathbf{x}) = -[\widehat{p}_{n,M}-p_{n+1}^-(\mathbf{x})][p_{n+1}^+(\mathbf{x})-\widehat{p}_{n,M}] \tag{17}$$
$$C_3(\mathbf{x}) = [p_n^+-p_{n+1}^+(\mathbf{x})]\pi_1(\mathbf{x}) \tag{18}$$

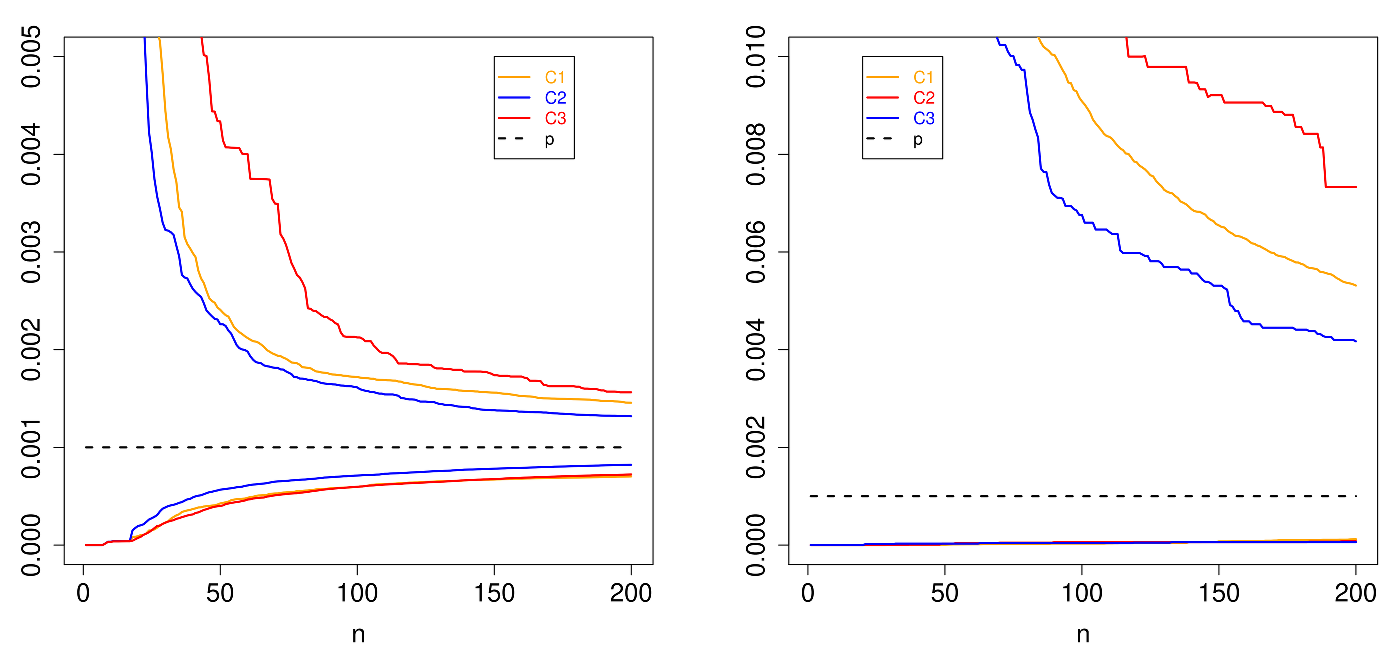

Figure 3: *Comparison of criteria C_1, C_2 and C_3*

4. Numerical applications

- A class of examples: in dimension d, let $\mathbf{X}=(X_1\dots,X_d)$ with $X_i\sim\Gamma(i+1,1)$ and

$$Z_d=\frac{X_1}{\sum_{i=1}^d X_i}\sim\mathcal{B}eta(2,(d+1)(d+2)/2-3) \tag{19}$$

Let $q_{d,p}$ be the p-order quantile of Z_d, and define

$$G(\mathbf{X})=Z_d-q_{d,p} \tag{20}$$

Then,

$$p=\mathbb{P}(G(\mathbf{X})\le 0)=\mathbb{P}(Z_d\le q_{d,p}) \tag{21}$$

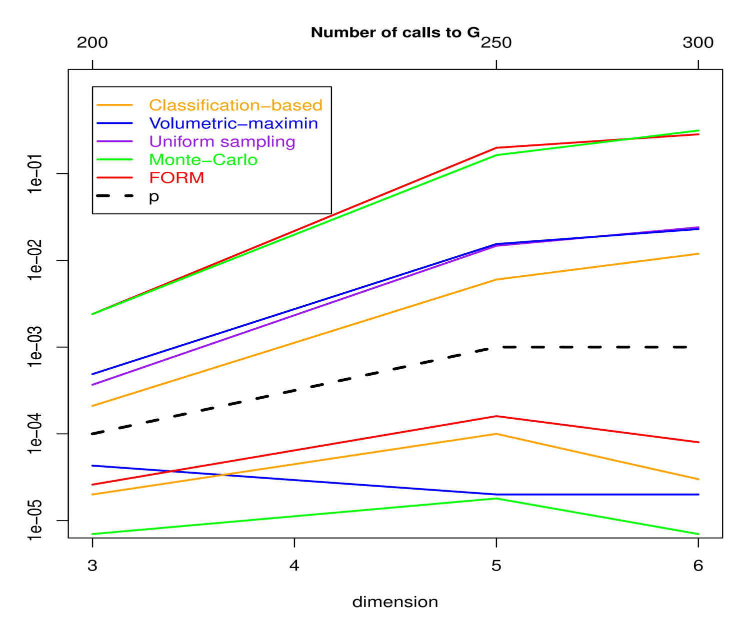

Figure 4: *Comparison of the exact bounds for geometric and classification-based criterion.*

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