Comparing conservative estimations of failure probabilities using sequential designs of experiments in monotone frameworks

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Abstract

A key point of structural reliability studies is to estimate the probability of an undesirable event. This estimation is made possible by using a numerical code that mimics the physical behavior of the studied phenomenon. The events considered are usually rare, occurring at the viewpoint of statistical learning theory. Structural Safety 26, 271-293

1. Industrial Context

Let $G : [0, 1]^d \rightarrow [0, 1]$ be a deterministic numerical model.

$\triangleright$ Uncertain inputs are represented by a random vector $\mathbf{X}$.

$\triangleright$ An indesirable event is described as $G(\mathbf{X}) < 0$.

$\triangleright$ The quantity of interest is $p = P(G(\mathbf{X}) \leq 0)$.

2. Monotonicity

$\triangleright$ A function $G$ is said monotonic if

$$G(\mathbf{x}) \leq G(\mathbf{y}) \quad \text{if} \quad \mathbf{x} \succeq \mathbf{y}.$$ (1)

$\triangleright$ Unless lost of generality one assume $G(\mathbf{0}) = 0$ and $G$ is increasing in all inputs. Let $X_i$ and define $f$, the cumulative density function of $X_i$. One transform the input as

$$X_i - \Phi^{-1}(F(X_i)).$$ (2)

$\triangleright$ Let $\mathbf{x}_i = (\mathbf{x}_1, \ldots, \mathbf{x}_d)$ obtained from any method of simulation and evaluated by $G$.

Considering

$$E_i = (\mathbf{x}_i, G(\mathbf{x}_i) \leq 0)$$ (3)

$$U_i = (\mathbf{x}_i, G(\mathbf{x}_i) > 0)$$ (4)

Two exact and deterministic bounds for $p$ can be obtained for all $n \geq 0$,

$$P(U \subseteq U^n) = \mathcal{P}_n \leq 1 - P(U \subseteq U^n)$$ (5)

with $U$ uniformly distributed on $U$.

3. Sequential Importance Sampling

$\triangleright$ Assume that at step $n$ of the exploration of input space $U$, the next point $x_n$ of the design is sampled from the importance distribution

$$x_n \sim f_{n-1} = \frac{f(x_n)}{P(U \subseteq U^n)}.$$ (6)

$\triangleright$ The idea is to choose $x_n$ which maximize a criterion $C$ such that $C(x_n)$ is near of $1$.

Denote $p^*_n(x) = P(U \subseteq U^n(x))$.

the contribution of $x$ for the reduction of the bounds. Where

$$U_{c}(x) = \{x \in U : y \in \mathbb{S}(\mathbb{E}(\mathbf{X})) x \leq y\},$$

$$U_{c, n}(x) = \{x \in U : y \in \mathbb{S}(\mathbb{E}(\mathbf{X})) x \leq y\}$$ (7)

Two classes of methods are proposed, the first one based on geometrical criterion and the second one based on classification tools.

$\triangleright$ The first criterion is the volumetric-maximin (V-Maximin) and is describe as follow

$$C(x) = \max \{p^*_n(x) - p_n(x)\}.$$ (8)

An alternative criterion called quick-maximin (Q-Maximin) is proposed

$$C(x) = \max \{p^*_n(x) - p_n(x)\}.$$ (9)

where

$$x = (x_1, \ldots, x_d) \in [0, 1]^d.$$ (10)

$\triangleright$ The second approach is based on classification tools. The problem to classify a new point is a problem of binary classification, which can be solved using monotonic neural networks. One proposes three criteria:

- $C_1(x) = (x - \mathbf{y}) (\mathbf{x} - \mathbf{y}) + (\mathbf{x} - \mathbf{y}) (\mathbf{x} - \mathbf{y})$ (11)

$\triangleright$ The comparison is made between $U_{c}(x)$ and $U_{c, n}(x)$.

$\triangleright$ A class of examples: in dimension $d$, let $X = (X_1, \ldots, X_n)$ with $X_i \sim \mathcal{U}(0, 1)$ and

$$Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

let $q_n$ be the $p$-order quantile of $Z_n$ and define

$$G(\mathbf{X}) = |Z_n - q_n|.$$ (12)

Then,$$

\begin{align*}
\mathcal{P}_n &\geq Z_n \leq q_n \\
\mathcal{P}_n &\geq Z_n \leq q_n.
\end{align*}

4. Numerical applications

$\triangleright$ Let $\mathbf{X} = (X_1, \ldots, X_n)$ with $X_i \sim \mathcal{U}(0, 1)$ and

$$Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

let $q_n$ be the $p$-order quantile of $Z_n$ and define

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References:


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