Introduction to Mixed Integer Nonlinear Programming

Luca Mencarelli

mencarelli@lix.polytechnique.fr

Group de Recherche Mascot-Num
Working Meeting “Handling Categorial and Continuous Data”
Amphithéâtre Hermite, Intitut Henri Poicaré, Paris

May 16, 2014

Outline of the Seminar

Introduction and Motivation

Convexity Issues

Correlated Problems

Complexity Issues

Basic Building Blocks

Algorithms and Softwares

The problem of the day

The general mixed integer nonlinear problem is

$$\text{MINLP} \left\{ \begin{array}{ll} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x,y) \leq 0 \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{array} \right.$$

- $f(x,y) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g(x,y) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ are smooth functions.
- $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^p$ are polytopes including bounds on the variables.

Extremely difficult: Combines challenges of handling nonlinearities with combinatorial explosion of integer variables [Belotti et al., 2013].

Extremely powerful: "The mother of all deterministic optimization problems" [Lee, 2008].

Real-world applications [Bonami, 2014]

Application	nonlinear	discrete
Portfolio optimization	Risk, utility, robustness	number of assets, min investment
[Bienstock, 1996, Bonami and Lejeune, 2009, Vielma et al., 2008]		
Chemical plant design	Chemical reactions	what to install
[Duran and Grossmann, 1986, Flores-Tlacuahuac and Biegler, 2007]		
Block Layout Design [Castillo et al., 2005]	Spatial constraints	what to layout
Networks with delays	Delay as function of traffic	Path, flows
[Boorstyn and Frank, 1977, Ameur and Ouorou, 2006]		
Location with stochastic services	Demands	location model
[Elhedhli, 2006]		
TSP with neighborhoods (Robotics) [Gentilini et al., 2013]	Definition of ngbh.	TSP

Real-world applications [Bonami, 2014]

Application	nonlinear	discrete
Petrochemical [Haverly, 1978]	Blending, pooling	Which process
Gaz/Water networks [Bragalli et al., 2011]	Pressure loss	Network topology
Nuclear Reactor reloading [Quist et al., 1999]	reactions	What to reload
Airplane trajectory optimization [Cafieri and Durand, 2013, Soler et al., 2013]	aerodynamics	waypoints, colision avoidance,...
Mixed Integer Optimal control [Sager, 2005, 2012]	DE	discrete controls
Countless more see for example [Belotti et al., 2013]	...	...

Convexity of Nonlinear Functions

MINLP $\begin{cases} \underset{x,y}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x,y) \le 0 \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{cases}$

Convex	Non-convex

MINLP

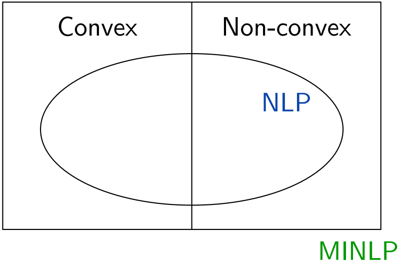
Definition. A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, iff for all couples of points $x^0, x^1 \in \mathbb{R}^n$, we have

$$f(x^1) \ge f(x^0) + \nabla f(x^0)^T (x^1 - x^0)$$

(In a slight abuse of notation) MINLP is convex iff $f(x,y)$ and $g(x,y)$ are convex functions, otherwise MINLP is nonconvex.

Nonlinear Programming

$$\text{NLP} \begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & g(x) \le 0 \\ & x \in X \end{cases}$$



- no integer variables, but challenge of handling nonlinearities.

- Convex NLP: all the minima all global minima (if strictly convex: only one minimum) and polynomial-time interior-point methods [Nesterov and Nemirovskii, 1994].

- Nonconvex NLP: find global solution is **NP**-hard (global quadratic optimization is already **NP**-hard [Sahni, 1974]).

MINLP $\neq$ NLP

- Convex NLP: all the minima all global minima and polynomial-time interior-point methods [Nesterov and Nemirovskii, 1994].

- A strictly convex NLP has at most one global solution, the same does not hold for MINLP.

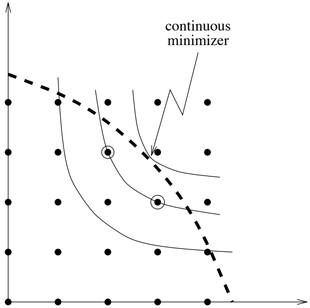
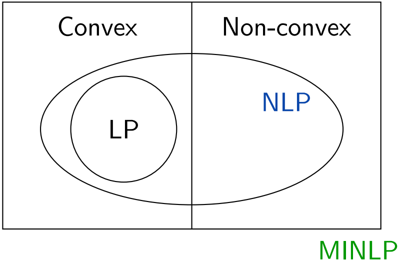


Figure: Optima for strictly convex MINLP [Leyffer, 1994].

Linear Programming

$$\text{LP} \begin{cases} \underset{x}{\text{minimize}} & a^T x \\ \text{subject to} & Ax \le c \\ & x \in X \end{cases}$$



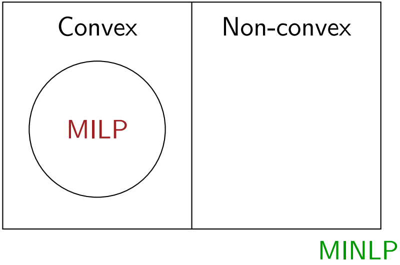
- no integer variables, no challenge of handling nonlinearities.

- polynomial-time methods: ellipsoid algorithm [Khachiyan, 1979] and Karmarkar's algorithm [Karmarkar, 1984].

- Simplex algorithm for LP, proposed by George Dantzig in 1947, is one of the Top Ten Algorithms of the 20th Century [Dongarra and Sullivan, 2000].

Mixed Integer Linear Programming

$$\text{MILP}\left\{\begin{array}{ll}\underset{x,y}{\text{minimize}} & a^T x + b^T y \\ \text{subject to} & Ax + By \le c \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p\end{array}\right.$$



- linear objective function and linear constraints, but combinatorial explosion of integer variables.

- **NP**-hard problem: no known polynomial-time algorithm (0-1 integer linear programming is **NP**-complete problems [Karp, 1972]).

- well-studied in literature since [Gomory, 1958] and very powerful algorithms (many commercial softwares).

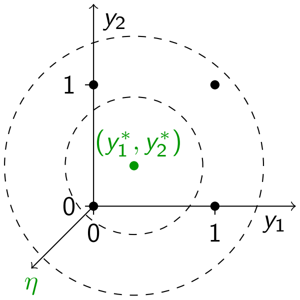
Convex MINLP $\neq$ MILP

The solution of MILP is an extreme point, however the same does not hold for convex MINLP:

$$\underset{y}{\text{minimize}} \sum_{i=1}^{p}\left(y_i - \frac{1}{2}\right)^2, \quad \text{subject to } y_i \in \{0,1\}, \quad i = 1,\ldots,p$$

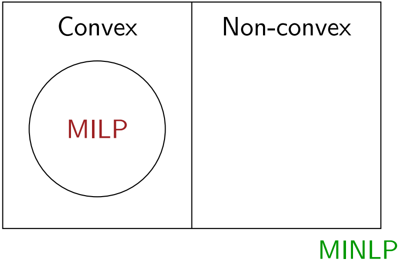
Trick: Introduce a new objective function (a dummy variable η) and a new constraint $f(x,y) \leq \eta$.

$$\text{MINLP}_\eta \begin{cases} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x,y) \leq \eta \\ & g(x,y) \leq 0 \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{cases}$$



Complexity Issues

$$\text{MINLP}_\eta \left\{ \begin{array}{ll} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x,y) \le \eta \\ & g(x,y) \le 0 \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{array} \right.$$



- MINLP is **NP**-hard (MILP as special case).

- In general undecidable [Jeroslow, 1973], even in "easy" case with "few" variables [De Loera et al., 2006].

- MINLP is a hot topic in optimization community: from [Leyffer, 1994] (first PhD thesis on convex MINLP) to IMA Hot Topics in 2012.

Assumptions and Hypotheses

Assumption 1. All the problem function are "perfectly" known, in terms of their mathematical expression and values.

Assumption 2. X and Y are nonempty compact convex sets defined by systems of linear inequality constraints.

Assumption 3. Functions f and g are twice continuously differentiable and convex.

Assumption 4. MINLP_η satisfies a constraint qualification condition.

- Assumptions 1: see next talk about Black Box MINLP.
- Assumptions 2 avoid undecidability problems.
- Assumptions 2 and 3 $\Longrightarrow$ we consider only convex MINLP_η.
- Assumptions 4: technical requirement for NLP machinery.

Moral: "the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity" [Rockafellar, 1993].

Auxiliary Problems

Relaxation in theory:

- optimize over a larger feasible region (ignore several constraints).
- compute a Lower Bound on the "real" minimum value.

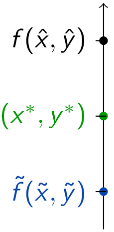
Definition. An optimization problem $\min\{\tilde{f}(z) : z \in \tilde{F}\}$ is a relaxation of $\min\{f(z) : z \in F\}$, iff $\tilde{F} \supset F$ and $\tilde{f}(z) \leq f(z)$ for all $z \in F$.

Relaxation in practice:

- relax integrality requirements: obtain an NLP.
- relax nonlinear constraints: obtain an MILP.

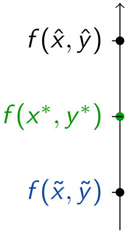
Upper bound:

- any feasible point provides an objective value greater (or equal) that the minimum one.



Relax Integrality: NLP$_{relax}$

$$
\text{NLP}_{relax} \left\{ \begin{array}{ll} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x,y) \leq \eta \\ & g(x,y) \leq 0 \\ & x \in X \\ & y \in Y \end{array} \right.
$$



- Relax integrality: from $y \in Y \cap \mathbb{Z}^p$ to $y \in Y \subset \mathbb{R}^p$.
- If MINLP$_\eta$ is convex then NLP$_{relax}$ is convex too (globally solvable).
- If $(\tilde{x}, \tilde{y})$ is optimal for NLP$_{relax}$ and feasible for MINLP$_\eta$, then it is also a minimum for MINLP$_\eta$.
- If $(\hat{x}, \hat{y})$ is feasible for MINLP$_\eta$ and $f(\hat{x}, \hat{y}) = f(\tilde{x}, \tilde{y})$, then it is also a minimum for MINLP$_\eta$.

Relax Convex Nonlinearities: MILP_{relax}

$$
\text{MILP}_{relax} \begin{cases} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x^k, y^k) + \left(\nabla f(x^k, y^k)\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \le \eta, \ \forall k \in \mathcal{K} \\ & g(x^k, y^k) + \left(\nabla g(x^k, y^k)\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \le 0, \ \forall k \in \mathcal{K} \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{cases}
$$

- Relax convex nonlinearities: supporting hyperplanes at points (x^k, y^k) for $k \in K$ (apply the definition of nonlinear convex function).
- Polyhedral (linear) relaxation of nonlinear convex constraints.
- Same relationships between solutions of MILP_{relax} and MINLP_{η}.

Relaxation in pictures

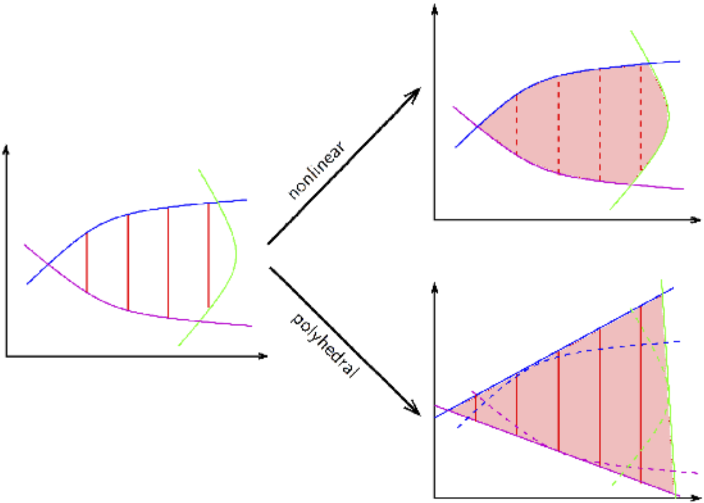
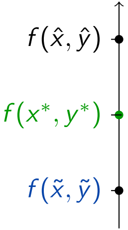


Figure: Nonlinear and polyhedral relaxation [Belotti et al., 2013].

Constraint Enforcement

$$
\text{MINLP}_\eta \left\{ \begin{array}{ll} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x,y) \le \eta \\ & g(x,y) \le 0 \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{array} \right.
$$



Goal: Exclude a solution $(\tilde{x},\tilde{y})$ of a relaxation, infeasible for of MINLP_η.

- Relaxation refinement: tighten the MILP_{relax} relaxation.
- Branching: exclude set of non-integer points from NLP_{relax}.
- Combinations of these two constraint enforcement approaches.

Constraint Enforcement: Refinement

Definition. A valid inequality is an inequality that is satisfied by all feasible solutions of MINLP_η. A cut is valid inequality that "cuts off" the current point $(\tilde{x}, \tilde{y})$ infeasible for MINLP_η.

Example. If $g(x, y) \leq 0$ convex and there is an index j such that $g_j(\tilde{x}, \tilde{y}) > 0$, then $(\tilde{x}, \tilde{y})$ is "cut off" by

$$g_j(\tilde{x}, \tilde{y}) + \nabla g_j(\tilde{x}, \tilde{y})^T \begin{pmatrix} x - \tilde{x} \\ y - \tilde{y} \end{pmatrix} \leq 0$$

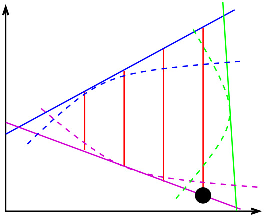


Figure: $(\tilde{x}, \tilde{y})$ infeasible for MINLP_η [Belotti et al., 2013].

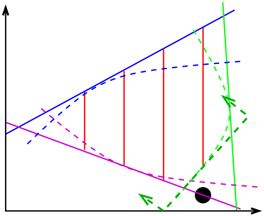


Figure: Valid inequality "cuts off" $(\tilde{x}, \tilde{y})$ [Belotti et al., 2013].

Constraint Enforcement: Branching

Goal: Exclude a fractional solution $(\tilde{x}, \tilde{y})$ of NLP_{relax}.

- Select fractional $\tilde{y}_i$ for some $i = 1, \ldots, p$.
- Create two new sub-problems by respectively adding:

$$y_i \leq \lfloor \tilde{y}_i \rfloor \quad \text{and} \quad y_i \leq \lceil \tilde{y}_i \rceil$$

- Solution to MINLP_η lies in one of the new subproblems.

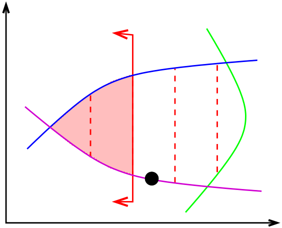


Figure: NLP subproblem with $y_i \leq \lfloor \tilde{y}_i \rfloor$ [Belotti et al., 2013].

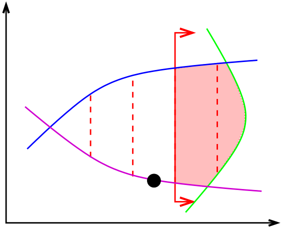
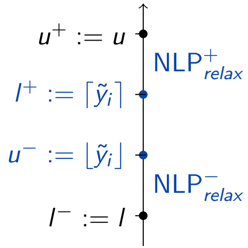


Figure: NLP subproblem with $y_i \geq \lfloor \tilde{y}_i \rfloor$ [Belotti et al., 2013].

Nonlinear Branch-and-Bound

$$
\text{NLP}_{relax} \begin{cases} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x,y) \le \eta \\ & g(x,y) \le 0 \\ & x \in X,\ y \in Y \\ & l \le y \le u \end{cases}
$$



- solve NLP_{relax} and find a fractional solution $(\tilde{x}, \tilde{y})$.

- introduce two new sub-problems NLP^+_{relax} and NLP^-_{relax} respectively with bounds $(l^+, u^+) := (l, u)$ and $(l^-, u^-) := (l, u)$:

$$
u_i^- := \lfloor \tilde{y}_i \rfloor \quad \text{and} \quad l_i^+ := \lceil \tilde{y}_i \rceil
$$

- nodes NLP^+_{relax} and NLP^-_{relax} correspond to the branching sub-problems.

Nonlinear Branch-and-Bound

- Iterate branching and create a tree.
- Let $(\hat{x}, \hat{y})$ be an upper bound of MINLP_η.

- Pruning rules:

Infeasibility: if a sub-problem is infeasible

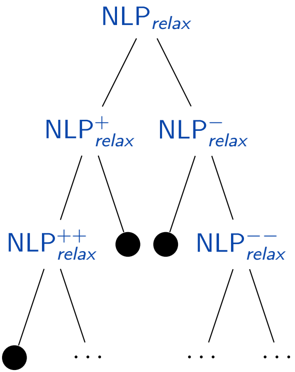
$\Rightarrow$ any NLP in its subtree is also infeasible.

Integrality: if $(\tilde{x}, \tilde{y})$ is an integral solution

(a) $f(\hat{x}, \hat{y}) < f(\tilde{x}, \tilde{y})$, then $(\hat{x}, \hat{y}) := (\tilde{x}, \tilde{y})$.
(b) no better feasible solution in sub-tree.

Dominance: if $f(\tilde{x}, \tilde{y}) \geq f(\hat{x}, \hat{y})$

$\Rightarrow$ no better integer solution in sub-tree.



Nonlinear Branch-and-Bound: Pseudocode

initialization: set $U := \infty$, and add NLP_{relax} to heap $\mathcal{H}$.
while $\mathcal{H} \neq \emptyset$ **do**
  Remove a sub-problem from the heap $\mathcal{H}$.
  Find a solution $(\tilde{x}, \tilde{y})$ to the current sub-problem.
  if current sub-problem is infeasible **then**
    Prune node by infeasibility.
  else if $f(\tilde{x}, \tilde{y}) \geq U$ **then**
    Prune node by dominance.
  else if $(\tilde{x}, \tilde{y})$ is integral **then**
    Update: $U := f(\tilde{x}, \tilde{y})$ and $(x^*, y^*) := (\tilde{x}, \tilde{y})$.
  else
    Branch on fractional variable.
    Create two sub-problems. Add them to $\mathcal{H}$.
  end if
end while

Nonlinear Branch-and-Bound

Theorem. All previous assumptions hold. Then Nonlinear Branch-and-Bound terminates at optimal solution (or indication of infeasibility) of MINLP_η after a finite number of iterations.

Open questions:

- How to select the branching variable (maximum fractional as bad as randomly selection [Achterberg et al., 2005]).
- How to select the next sub-problem to be solved.
- Warm-starting of NLP solver.

Goal: Minimize size of Branch-and-Bound tree (dimension of heap $\mathcal{H}$).

Strategy: Find good upper bound and increase lower bound quickly.

Outer Approximation [Duran and Grossmann, 1986]

$$
\text{MILP}_{relax} \begin{cases} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & f(x^k,y^k) + (\nabla f(x^k,y^k))^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \leq \eta, \ \forall k \in \mathcal{K} \\ & g(x^k,y^k) + (\nabla g(x^k,y^k))^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \leq 0, \ \forall k \in \mathcal{K} \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{cases}
$$

- Tighten MILP_{relax} by iteratively adding supporting hyperplanes (valid inequalities).
- Evaluate convex functions only at "integer" points (x^k, y^k) for all $k \in \mathcal{K}$.
- MILP machinery: more than 50 years of experience in theory and practice.

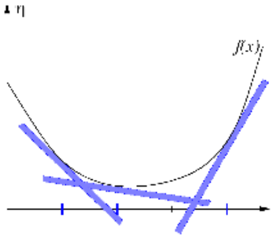


Figure: MILP_{relax} [Leyffer, 2013].

Outer Approximation: NLP sub-problem

$$\text{NLP}_{fix} \begin{cases} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & g(x, y) \leq 0 \\ & x \in X \\ & y = \hat{y} \end{cases} \qquad \text{NLP}_{feas} \begin{cases} \underset{x}{\text{minimize}} & \sum_{j=1}^{m} g_j^+(x, \hat{y}) \\ \text{subject to} & x \in X \end{cases}$$

- NLP_{fix} is feasible $\Longrightarrow$ upper bound to MINLP_{η}.
- NLP_{fix} is infeasible $\Longleftrightarrow$ strictly positive optimal value of NLP_{feas}.
- NLP_{fix} is infeasible $\Longrightarrow$ nonlinear solvers provide a solution to NLP_{feas}.

Lemma. Supporting hyperplanes at $(\hat{x}, \hat{y})$ are valid inequalities for MINLP_{η}. If NLP_{fix} is infeasible, then supporting hyperplanes are valid cut w.r.t. $(\hat{x}, \hat{y})$.

Outer Approximation Algorithm

$$
\text{MILP}_{relax} \begin{cases} \underset{x,y,\eta}{\text{minimize}} & \eta \\ \text{subject to} & \eta \le U^k - \varepsilon \\ & f(x^k,y^k) + \left(\nabla f(x^k,y^k)\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \le \eta, \ \forall x^k \in \mathcal{X}^k \\ & g(x^k,y^k) + \left(\nabla g(x^k,y^k)\right)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} \le 0, \ \forall x^k \in \mathcal{X}^k \\ & x \in X \\ & y \in Y \cap \mathbb{Z}^p \end{cases}
$$

- $\mathcal{X}^k \subset \mathcal{X} := \{(\hat{x}^k, \hat{y}^k) \in X \times Y \cap \mathbb{Z}^p : (\hat{x}^k, \hat{y}^k)$ solutions to NLP_{fix} or NLP_{feas} for integer assignment $\hat{y} = y^k\}$.
- boundedness of $Y \cap \mathbb{Z}^p \implies$ boundedness of $\mathcal{X}$ (and of $\mathcal{X}^k$).
- upper bound $U^k := \min_{j \le k}\{f(\hat{x}^k, \hat{y}^k) : \text{NLP}_{fix}$ is feasible$\}$.

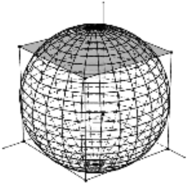
Outer Approximation: Pseudocode

data: starting integer point (x^0, y^0) and tolerance $\varepsilon > 0$.
initialization: set $U^{-1} := \infty$, $\mathcal{X}^{-1} = \emptyset$ and $k = 0$.
repeat
  Solve NLP_{fix} or NLP_{feas} with $\hat{y} = y^k$: solution $(\hat{x}^k, \hat{y}^k)$.
  if NLP_{fix} is feasible and $f(\hat{x}^k, \hat{y}^k) < U^{k-1}$ **then**
    Update best point $(x^*, y^*) = (\hat{x}^k, \hat{y}^k)$ and $U^k = f(\hat{x}^k, \hat{y}^k)$.
  else
    Set $U^k = U^{k-1}$.
  end if
  Add supporting hyperplanes about $(\hat{x}^k, \hat{y}^k)$ to MILP_{relax}:
  $\mathcal{X}^k = \mathcal{X}^{k-1} \cup \{k\}$.
  Solve MILP_{relax}: solution (x^{k+1}, y^{k+1}) and set $k = k + 1$.
until MILP_{relax} is infeasible.

Worst case of OA [Hijazi et al., 2014]

Theorem. All previous assumptions hold. Then Outer Approximation terminates at optimal solution (or indication of infeasibility) of MINLP_η after a finite number of iterations.

$$P \begin{cases} \underset{y}{\text{minimize}} & 0 \\ \text{subject to} & \displaystyle\sum_{i=1}^{p}\left(y_i - \frac{1}{2}\right)^2 \leq \frac{p-1}{4} \\ & y \in \{0,1\}^p \end{cases}$$



Lemma. Outer Approximation cannot cut more than one vertex of the hypercube: MILP_{relax} feasible for any $k < 2^n$.

Corollary. Outer Approximation Algorithm takes 2^n iterations (each of them requires solving a MILP_{relax}) to check infeasibility of nonlinear integer problem P.

LP/NLP-based BB [Quesada and Grossman, 1992]

Aim: avoid solving expensive many MILP_{relax}'s.

- Start solving MILP_{relax} by Branch-and-Bound.
- If an integer solution $(\tilde{x}, \tilde{y})$ is found

$\Rightarrow$ solve NLP_{fix} with $\hat{y} = \tilde{y}$, get $(\hat{x}, \hat{y})$.

- Add supporting hyperplanes about $(\hat{x}, \hat{y})$ to single Branch-and-Bound tree.
- Continue by solving MILP_{relax} problem.
- Iterate until lower bound $\geq$ upper bound.
- Never prune by integer feasibility.

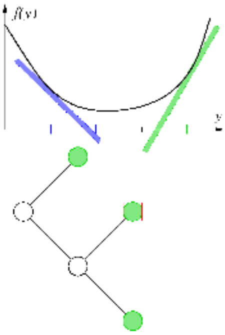


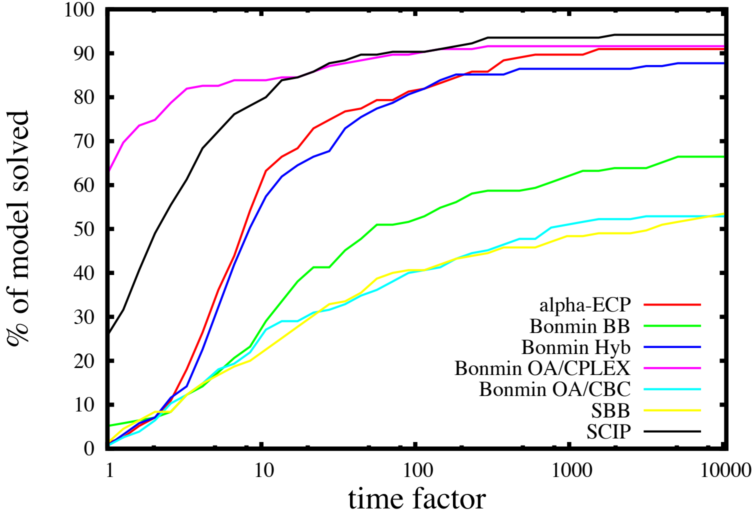
Figure: LP/NLP-based BB [Leyffer, 2013].

Theorem. All previous assumptions hold. Then LP/NLP-based Branch-and-Bound terminates at optimal solution (or indication of infeasibility) of MINLP_{η} after a finite number of iterations.

Convex MINLP Softwares [Bonami, 2014]

Solver	Reference	Algorithm(s)
Dicopt		OA
MINLP_BB	[Leyffer, 1998]	NLP BB
SBB	[Bussieck and Drud, 2001]	NLP BB
α-ECP	[Westerlund and Lundqvist, 2005]	ECP (variant of OA)
Bonmin	[Bonami et al., 2008]	NLP BB, OA, LP/NLP
FilMINT	[Abhishek et al., 2010]	LP/NLP
KNITRO	[Byrd et al., 2006]	NLP BB, LP/NLP
SCIP	[Vigerske, 2013]	LP/NLP

Convex MINLP Softwares [Bonami, 2014]



Black Box MINLP: Derivative-free approaches

- objective function values are too expensive to compute.
- objective function values are determined via simulation.

Derivative-free methods:

- Direct search: locally sampling objective function along a grid.
- Evolutionary methods: random mutations and survival of the fittest.

Software:

- NOMAD (Nonsmooth Optimization by Mesh Adaptive Direct Search) [Audet et al., 2009], [Le Digabel, 2011].

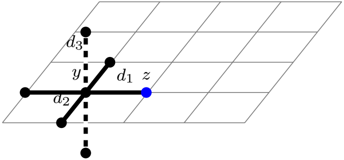


Figure: Local direct search [Liuzzi et al., 2014].

Conclusions: Convex MINLP

- Extremely powerful and difficult: very challenging problem and many real-world applications.

- Convexity and completely knowledge of functions are critical assumptions.

- Key building blocks: NLP and MILP relaxations with constraint enforcement (branching and refinement).

- Basic algorithms for convex MINLP: Nonlinear Branch-and-Bound and Outer Approximation.

- One step beyond: LP/NLP-based Branch-and-Bound (Outer Approximation embedded in a single tree).

- MINLP softwares in a nutshell with a comparison picture: Outer Approximation performs relatively better.

- Reformulation is a very important ingredient in every optimization recipe: exploit structure of your problem.

References

K. Abhishek, S. Leyffer and J.T. Linderoth
FilMINT: An Outer-Approximation-based Solver for Convex Mixed-Integer Nonlinear Programs
INFORMS Journal on Computing, 22(4):555-567, 2010

T. Achterberg, T. Koch, A. Martin
Branching Rules Revisited
Operations Research Letters, 33(1):42-54, 2004

W.B. Ameur and A. Ouorou
Mathematical Models of the Delay Constrained Routing Problem
Algorithmic Operations Research, 1(2):94-103, 2006

C. Audet, S. Le Digabel and C. Tribes
NOMAD User Guide
Technical Report G-2009-37, Les Cahiers du GERAD, 2009

P. Belotti, C. Kirches, S. Leyffer, J. Linderoth, J. Luedtke and A. Mahajan
Mixed-Integer Nonlinear Optimization
Acta Numerica, 22:1-131, 2013

D. Bienstock
Computational Study of a Family of Mixed-Integer Quadratic Programming Problems
Mathematical Programming, 74(2):121-140, 1996

References

P. Bonami
Mixed Integer Nonlinear Programming Algorithms
Talk, 15ème Congrès Annuel de la Société Française de Recherche Opérationnelle et d'Aide à la Décision (ROADEF), Session Plénière, Université de Bordeaux, 2014

P. Bonami and M. Lejeune
An Exact Solution Approach for Integer Constrained Portfolio Optimization Problems Under Stochastic Constraints
Operations Research, 57(3):650-670, 2009

P. Bonami, L.T. Biegler, A.R. Conn, G. Cornuéjols, I.E. Grossmann, C.D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya and A. Wächter
An Algorithmic Framework for Convex Mixed Integer Nonlinear Programs
Discrete Optimization, 5(2):186-204, 2008

R. Boorstyn and H. Frank
Large-Scale Network Topological Optimization
IEEE Transactions on Communications, 25(1):29-47, 1977

C. Bragalli, C. D'Ambrosio, J. Lee, A. Lodi and P. Toth
On the Optimal Design of Water Distribution Networks: A Practical MINLP Approach
Optimization and Engineering, 13(2):1-28, 2012

References

M.R. Bussieck and A. Drud
SBB: A New Solver for Mixed Integer Nonlinear Programming
Talk, OR 2001, Section "Continuous Optimization", Duisburg, 2001

R.H. Byrd, J. Nocedal and R.A. Waltz
KNITRO: An Integrated Package for Nonlinear Optimization
Large Scale Nonlinear Optimization, pages 35-59, Springer Verlag, 2006

S. Cafieri and N. Durand
Aircraft Deconfliction with Speed Regulation: New Models from Mixed-Integer Optimization
Journal of Global Optimization, 58(4):613-629, 2014

I. Castillo, J. Westerlund, S. Emet and T. Westerlund
Optimization of Block Layout Design Problems with Unequal Areas: A Comparison of MILP and MINLP Optimization Methods
Computers and Chemical Engineering, 30(1):54-69, 2005

J.A. De Loera, R. Hemmecke, M. Köppe, and R. Weismantel
Integer Polynomial Optimization in Fixed Dimension
Mathematics of Operations Research, 31(1):147-153, 2006

J. Dongarra and F. Sullivan
Top Ten Algorithms of the Century
Computing in Science and Engineering, 2(1):22-23, 2000

References

M.A. Duran and I. Grossmann
An Outer-Approximation Algorithm for a Class of Mixed-Integer Nonlinear Programs
Mathematical Programming, 36(3):307-339, 1986

S. Elhedhli
Service System Design with Immobile Servers, Stochastic Demand, and Congestion
Manufacturing & Service Operations Management, 8(1):92-97, 2006

A. Flores-Tlacuahuac and L.T. Biegler
Simultaneous Mixed-Integer Dynamic Optimization for Integrated Design and Control
Computers and Chemical Engineering, 31(5-6):648-656, 2007

I. Gentilini, F. Margot and K. Shimad
The Traveling Salesman Problem with Neighborhoods: MINLP Solution
Optimization Methods and Software, 28(2):364-378, 2013

R.E. Gomory
Outline of an Algorithm for Integer Solutions to Linear Programs
Bulletin of the American Mathematical Society 64(5):275-278., 1958

C.A. Haverly
Studies of the Behavior of the Recursion for the Pooling Problem
ACM SIGMAP Bulletin, 25:19-28, 1978

References

H. Hijazi, P. Bonami and A. Ouorou
An Outer-Inner Approximation for Separable Mixed-Integer Nonlinear Programs
INFORMS Journal of Computing, 26(1):31-44, 2014

R.C. Jeroslow
There Cannot Be any Algorithm for Integer Programming with Quadratic Constraints
Operations Research, 21(1):221-224, 1973

N. Karmarkar
A New Polynomial-time Algorithm for Linear Programming
Combinatorica, 4(4):373-395, 1984

R.M. Karp
Reducibility Among Combinatorial Problems
Complexity of Computer Computations, pages 85-103, New York, Plenum, 1972

L.G. Khachiyan
A Polynomial Algorithm in Linear Programming
Doklady Akademiia Nauk SSSR 244, 1093-1096, 1979 (English translation: *Soviet Mathematics Doklady, 20(1):191-194*, 1979)

J. Lee
How We Participate in Open Source Agreements
URL: `http://ibm.co/1nPMkAM`, 2008

References

S. Leyffer
Deterministic Methods for Mixed Integer Nonlinear Programming
PhD Thesis, University of Dundee, 1993

S. Leyffer
Integrating SQP and Branch-and-Bound for Mixed Integer Nonlinear Programming
University of Dundee Numerical Analysis Report NA/182, 1998

S. Leyffer
Mixed-Integer Nonlinear Optimization: Applications, Algorithms, and Computation
PhD Course, Graduate School in Systems, Optimization, Control and Networks, Université Catholique de Louvain, 2013

G. Liuzzi, S. Lucidi and F. Rinaldi
Derivative-free Methods for Mixed-Integer Constrained Optimization Problems
URL: `http://www.optimization-online.org/DB_HTML/2014/03/4269.html`, 2014

Y. Nesterov and A.S. Nemirovskii
Interior-Point Polynomial Algorithms in Convex Programming
Society for Industrial and Applied Mathematics (SIAM), 1994

References

I. Quesada and I.E. Grossmann
An LP/NLP based Branch-and-Bound Algorithm for Convex MINLP Optimization Problems
Computers and Chemical Engineering, 16(10-11):937-947, 1992

A.J. Quist, R. van Gemeert, J.E. Hoogenboom, T. Ílles, C. Roos and T. Terlaky
Application of Nonlinear Optimization to Reactor Core Fuel Reloading
Annals of Nuclear Energy, 26(5):423-448, 1999

R.T. Rockafellar
Lagrange Multipliers and Optimality
SIAM Review, 35(2):183-238, 1993

S. Sahni
Computationally Related Problems
SIAM Journal on Computing , 3(4):262-279, 1974

S. Sager
Numerical Methods for Mixed-Integer Optimal Control Problems
PhD Thesis, Interdisciplinary Center for Scientific Computing, Universität Heidelberg, 2005

S. Sager
A Benchmark Library of Mixed-Integer Optimal Control Problems
Mixed Integer Nonlinear Programming, pages 631-670, Springer Verlag, 2012

References

M. Soler, P. Bonami, A. Olivares and E. Staffetti
Multiphase Mixed-Integer Optimal Control Approach to Aircraft Trajectory Optimization
Journal of Guidance, Control, and Dynamics, 36(5):1267-1277, 2013

J.P. Vielma, S. Ahmed and G. Nemhauser
A Lifted Linear Programming Branch-and-Bound Algorithm for Mixed Integer Conic Quadratic Programs
INFORMS Journal on Computing, 20(3):438-450, 2008

S. Vigerske
Decomposition in Multistage Stochastic Programming and a Constraint Integer Programming Approach to Mixed-Integer Nonlinear Programming
PhD thesis, Humboldt-Universität zu Berlin, 2012

T. Westerlund and K. Lundqvist
Alpha-ECP, Version 5.101: An Interactive MINLP-Solver Based on the Extended Cutting Plane Method
Technical Report 01-178-A, Process Design Laboratory, Abo Akademi University, 2008