Calibration of a PDE system for thermal regulation of an aircraft cabin

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Outline

1 Context

2 Calibration from experimental data

3 Meta model strategy

4 Summary & challenges
PDE calibration for aircraft cabin thermal regulation

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1. Context

2. Calibration from experimental data

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General context of thermal regulation

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External environment

Avionic bay

Equipment
Provide thermal comfort & cabin pressurization for crew / passengers

Thermal control of electric cores or highly dissipative equipment of avionic bay
PDE calibration for aircraft cabin thermal regulation

General context of thermal regulation

- Provide thermal comfort & cabin pressurization for crew / passengers
- Thermal control of electric cores or highly dissipative equipment of avionic bay
Installation of equipment in avionic bay requires the specification of equipment thermal environment

**Figure**: Aircraft & Equipment - Avionic bay

- Need to provide convection coefficients around the equipment...
- ... For a robust equipment conception
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Topics of the presentation

**PHASE I**
Model Calibration

**PHASE II**
Thermal Analyses

Two phases:
- 1/ PDE parameter estimation
- 2/ Phenomenon study with parametrized PDE
Two phases:

- **1/ PDE parameter estimation**
- **2/ Phenomenon study with parametrized PDE**
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Modelling

- **(simplified) Thermal exchange modelling** (Navier Stokes equations)

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0 \\
\frac{\partial (\rho C_p T)}{\partial t} + \nabla \cdot (u \rho C_p T) &= \nabla \cdot (k \nabla T) \\
\frac{\partial (\rho u)}{\partial t} + (u \nabla) u + \nabla p &= \mu \Delta u + \rho g
\end{align*}
\]

Boundary Conditions:
\[
\begin{align*}
u &= u_0(M) \text{ with turbulence model } \text{RANS}(\tau) \\
\phi &= h_C (T - T_{\text{Skin}})
\end{align*}
\]

\(\rho\) = air density, \(u\) = air speed, \(T\) = temperature, \(\tau\) = turb. rate, \(h_C\) = heat transf. coef., \(T_{\text{Skin}}\) = skin temp.
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\[
\rho = \text{air density}, \quad u = \text{air speed}, \quad T = \text{temperature}, \quad \tau = \text{turb. rate}, \quad h = \text{heat transf. coef.}, \quad T_{Skin} = \text{skin temp.}
\]

⇒ Lack of knowledge on \(\tau, h_C\) and \(T_{Skin}\)!
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\(\Rightarrow h_C\) should be estimated
\(\Rightarrow \tau\) and \(T_{\text{Skin}}\) are subjected to uncertainties
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\(\rho=\) air density, \(u=\) air speed, \(T=\) temperature, \(\tau=\) turb. rate, \(h=\) heat transf. coef., \(T_{\text{Skin}}=\) skin temp.

\(\Rightarrow h_C\) should be **estimated**

\(\Rightarrow \tau\) and \(T_{\text{Skin}}\) are subjected to **uncertainties**

- **Input/Output model view**

  Equation & Boundary Conditions induce an Input/Output system

  \(\mathcal{H}((\tau, T_{\text{Skin}}), h_C)\).

  In particular, the post-processing providing convection coefficients is some function \(h((\tau, T_{\text{Skin}}), h_C)\).
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Question?

How to estimate $h_C$ in presence of uncertainties ($\tau$, $T_{Skin}$)?

- We need additional information (reference measures, experimental data, etc.)
- How to model the uncertainties?
- How to take into account uncertainties in identification procedures?
Outline

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4. Summary & challenges
Experiments

**Figure**: Flight test - Chamber test

- **Principle:**
  At a fixed environmental condition, one can measure convection coefficients $C_{i}^{obs}$ around the equipment.
  - Flight tests / Chamber tests
  - Few sensors are used
Experiments

**Figure**: Flight test - Chamber test

**Principle:**
At a fixed environmental condition, one can measure convection coefficients \( C_{i}^{obs} \) around the equipment.

- Flight tests / Chamber tests
- Few sensors are used

Finally, one gets a very precious database \( (C_{i}^{obs}) \) for \( i = 1, \ldots, N \) with \( N \) limited!
Summary

We have two ingredients:

- We can compute convection coefficients of the equipment from Navier Stokes equations
  \[ C^{comp} = h((\tau, T_{Skin}), h_C) \]

- Experimental database
  \[(C_{i}^{obs})_{i=1,...,N}\]
Summary

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Remark: a single run of \( h \) may take several hours (\( \sim 6 \) hours !)
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Remark: a single run of \( h \) may take several hours (\( \sim 6 \) hours !)

Q : How to estimate \( h_C \) from the experimental database ?
Mathematical formalization

- **Variable of interest** (induced by a PDE system)
  We call a variable of interest any quantity obtained by a post-processing of some PDE equations resolution. It takes the form
  \[ h(X, \theta) \]
  where
  - \( X \in (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), P_x) \) is a random vector representing the uncertainties
  - \( \theta \in \mathbb{R}^k \) is the vector of parameters to identify
  (in our application: \( X = (\tau, T_{Skin}) \in (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P_x) \) and \( \theta = h_C \in \mathbb{R} \))

- **Experimental/Reference data** (also called **Learning data**)
  It is a set of points:
  - \((z_i, Y_i)_{i=1,...,N} \) → if \( h(X, \theta) \) is a field \( z \mapsto h(X, \theta)[z] \)
  - \((Y_i)_{i=1,...,N} \) → if \( h(X, \theta) \) is scalar
  (Remark: a priori, **There is not** a model linking the observation \( Y \) and the simulation \( h(X, \theta) \). For instance, we don’t have the regression framework
  \[ Y = h(X, \theta) + \varepsilon \]
  where \( \varepsilon \) is the model error. Indeed, we **don't have** joint information \((X_i, Y_i) \) ! )
There are two calibration methods depending on the nature of the variable of interest $h(X, \theta)$, scalar or field.
Calibration method I (case for fields)

- **Least Squares principle:**
  Find parameters $\theta \in \mathbb{R}^k$ which minimize the quantity

$$
\mathcal{J}(\mathbf{X}, \theta) = \sum_{i=1}^{N} (Y_i - h(\mathbf{X}, \theta)[z_i])^2
$$

- **Remark !:**
  the function $\theta \mapsto \mathcal{J}(\mathbf{X}, \theta)$ to minimize is random (due to uncertainties $\mathbf{X}$)!
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   $\Rightarrow$ Classical least squares methods are infeasible ...
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  \( \Rightarrow \) Classical least squares methods are infeasible ...

- **Issue: Stochastic Optimization**

  **Principle:** Minimize a quantity \( \rho(\mathcal{J}(X, \theta)) \) (deterministic)
  - Mean : \( \rho(\mathcal{J}(X, \theta)) = \mathbb{E}_X(\mathcal{J}(X, \theta)) \)
  - Variance : \( \rho(\mathcal{J}(X, \theta)) = \text{Var}_X(\mathcal{J}(X, \theta)) \)
  - Mixed : \( \rho_\lambda(\mathcal{J}(X, \theta)) = \mathbb{E}_X(\mathcal{J}(X, \theta)) + \lambda \sqrt{\text{Var}_X(\mathcal{J}(X, \theta))} \)
  - etc.
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Calibration method I (case for fields)

Illustration

\[ \theta \mapsto \rho_\lambda(J(X, \theta)) = \mathbb{E}_X(J(X, \theta)) + \lambda \sqrt{\text{Var}_X(J(X, \theta))} \text{ for different } \lambda > 0 \]

(deterministic function \( \Leftrightarrow \theta \mapsto J(X_{\text{nom}}, \theta) \), where \( X_{\text{nom}} \) is the nominal value of \( X \))
Calibration method I (case for fields)

- Stochastic/Robust Optimization
  - Large literature
  - Practical algorithms
    Need practical and efficient algorithms ...
Recall the framework:

- We have observations \((Y_i)_{1,...,N}\)
- We get a scalar output \(h(X, \theta)\) after a post-processing of a PDE system
Calibration method II (case for scalar outputs)

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- Estimation method:


**Principle:**

Find parameters \(\theta \in \mathbb{R}^k\) which minimize "a distance" between the **empirical distribution** of the \(Y_i\)’s and the **simulated distribution** of the random variable \(h(X, \theta)\) (based on a simulated sample \(h(X_1, \theta), ..., h(X_m, \theta)\), where \(X_1, ..., X_m\) are \(m\) simulations of the uncertainty \(X\)).
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Calibration method II (case for scalar outputs)

- Recall the framework:
  - We have observations \((Y_i)_1, \ldots, N\)
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- Example: Maximum-Likelihood based method

Find \(\theta\) minimizing

\[
\mathcal{J}(\theta) = -\sum_{i=1}^{N} \log \left( \sum_{j=1}^{m} K_b(Y_i - h(X_j, \theta)) \right), \quad \text{with} \quad K_b(y) = \frac{1}{\sqrt{2\pi b}} e^{-y^2/2b^2}
\]
Calibration method II (case for scalar outputs)

Theoretical results of the estimator $\hat{\theta}_{N,m}$ where

$$\hat{\theta}_{N,m} = \text{Argmin} \sum_{\theta \in \Theta} N \log \left( \sum_{i=1}^{m} K_b \left( Y_i - h(X_j, \theta) \right) \right)$$
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Theorem (Consistency) [Rachdi2012]

Denote by $f_{\theta}^x$ the density function of $h(X, \theta)$ and $\theta^*$ by

$$\theta^* = \text{Argmin}_{\theta \in \Theta} -\mathbb{E}(\log(f_{\theta}^x)(Y)) \quad \text{(unknown target)}.$$ 

Under technical conditions, $\exists$ constants $c_1, c_2, c_3, a_1, a_2$ such that $\forall 0 < \tau < 1/2$, with probability at least $1 - 2 \tau$

$$\|\hat{\theta}_{N,m} - \theta^*\|^2 \leq c_1 \sqrt{\frac{\log(a_1 \tau^{-1})}{N}} + \frac{c_2 \sqrt{\log(a_2 \tau^{-1})}}{\sqrt{m}} + c_3 m^{1/10}.$$
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$\Rightarrow$ the right hand side is not the rate of convergence! ... but ensure the consistency.
Calibration method II (case for scalar outputs)

Theoretical results of the estimator \( \hat{\theta}_{N,m} \) where

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\]

\( \Rightarrow \) the right hand side is not the rate of convergence ! ... but ensure the consistency.

**Theorem (Central Limit Theorem)**

*In progress!*
Simulation of \( h \) is limited!

- **Calibration may be very greedy ...**
  Both calibration methods may need several computations of \( h \) involving new PDE system resolutions.
  - In most of our applications, one run of \( h \) (i.e numerical resolution + post-processing) \( \sim 6 \) hours
  - So for 50 calibration algorithm iterations, we have to wait \( \sim 13 \) days!

- **Strategy adopted:**
  Replace the CPU time expensive model \( h(X, \theta) \) by a mathematical approximation (analytical) \( \tilde{h}(X, \theta) \), very cheap to evaluate.
PDE calibration for aircraft cabin thermal regulation

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Meta model strategy

A well adopted strategy (among others...):

- **Sample, Build, Validate and Replace**

Different types of meta models

- **Regression-based**: (Neural network, Polynomial Chaos, Least squares, etc.)
- **Interpolation-based**: (Radial Basis Functions, Gaussian processes/Kriging, etc.)

Calibration methods only involve the metamodel, i.e one calibrates the metamodel! (no more the PDE system...)
PDE calibration for aircraft cabin thermal regulation

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Summary: global process of thermal analysis

- CFD model setup
  Must be valid on the parameter set range

- DOE (Design Of Experiment) generation
  Choice of parameters, observables
  Experts knowledge

- Post-processing
  Observables data output

- Results database build up

- HPC

- THERMAL ANALYSES
  convec. coef, temperatures, etc.

- Meta model setup
  Meta modeling methods (Regression, Krigging, RBF...)

- Calibration

- Meta model integration in a modelling software (Dymola, etc.)
Conclusions & Issues

- Asymptotic study of the estimator $\hat{\theta}_{N,m}$
- Mathematical study of calibration procedures induced by the Stochastic Optimization of $\theta \mapsto J(X, \theta)$
- Quantify the robustness of equipment specification when considering the uncertainties
- Improve existing metamodel-based algorithms (adaptive metamodelling, on-line refinement, etc.)
- HPC capabilities for metamodel constructions
- Facilitate metamodels exportation (distribution to suppliers, etc.)
- Extend the method for Multi-Fidelity learning data (varying mesh size, etc.)
Thank you for your attention!


A. Shapiro, D. Dentcheva, A. Ruszczynski (2009), Lectures on Stochastic Programming, MPS-SIAM Series on Optimization

A. Ben-Tal, L. El Ghaoui, A. Nemirovski (2009), Robust Optimization, Princeton Series in Applied Mathematics